A Numerical Scheme to Solve Integro-Differential Equations in the Dynamic Reliability Field

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1 Introduction - Notations

A system is considered, which evolves in time. At time t, its state is described by its "physical" state denoted by I_t and by some "environmental" conditions, such as temperature, pressure or so on, and denoted by X_t. The "physical" state I_t is assumed to take its values in a finite state-space E whereas the "environmental" conditions X_t take their values in $\mathbb{R}^d$. For i, j in E, the transition rate at time t from state i to state j depends on the value (say x) of the "environmental" conditions X_t and is denoted by $a(i, j, x)$. The evolution of the environmental conditions is described by a set of differential equations, which depends on the state of the item. More precisely, given that the system is in state $I_t = i$ for all $t \in [a, b]$, then X_t fulfils the following equation: $\frac{dX_t}{dt} = v(i, X_t)$ for all $t \in [a, b]$, where v is an application from $E \times \mathbb{R}^d$ to $\mathbb{R}^d$. This is the general context of dynamic reliability.

In such a field, the process (X_t) is usually assumed to be continuous so that no jump is allowed for the environmental conditions. In the present paper, this assumption of continuity is relaxed and the process (X_t) may here jump simultaneously with the process (I_t). More precisely, when the process (I_t) jumps from state i to state j, we assume that the process (X_t) belongs to $[y, y+dy[$ with probability $\mu(i, j, x)(dy)$ at the jump time, given that the environmental conditions were x just before the jump. In case $\mu(i, j, x)(dy) = \delta_x(dy)$ (for all i, j, x), the process (X_t) is continuous and our model then meets with the usual one. Our model however includes a lot of other interesting cases. For instance, suppose that there is a single environmental condition with values in $\mathbb{R}_+$ and suppose that $v \equiv 1$. Then, if $\mu(i, j, x)(dy) = \delta_0(dy)$ for all i, j, x, the process (I_t) is a semi-Markov process. If $\mu(i, j, x)(dy) = \delta_x(dy)$ for all i, j, x, then $X_t = X_0 + t$ and the process (I_t) is a non-homogenous Markov process. More generally, it can be shown that interacting semi-Markov processes are particular cases of our model: for d interacting semi-Markov processes, the environmental conditions have to be taken in $\mathbb{R}^d_+$ and we may then choose $\mu(i, j, x)(dy) = \delta_{x(1),\ \ldots,x(k-1),0,x(k+1),\ \ldots,x(d)}(dy)$ for some suitable k depending on i and j.

Our goal here is to propose a new numerical method to compute the distribution $\pi(t)(i, dx)$ of the process (I_t , X_t) at time t. Let us recall that such a distribution is analytically calculable in only very simple cases, so that generally only numerical computations are reachable. One of the most up-date methods for such numerical computations appeals to Monte-Carlo simulation where the authority of P.E. Labeau in the dynamic reliability field is well-known. We here propose a new method, based on a numerical scheme which converges towards the distribution $\pi(t)(i, dx)$. With that aim, we first characterize the distribution $\pi(t)(i, dx)$ as the single solution of a set of integro-differential equations (the Chapman-Kolmogorov equations) and we then derive a numerical scheme based on the finite volumes method. The singleness of the solution of the Chapman-Kolmogorov equations is here crucial to show the convergence of the numerical scheme towards the right solution. Finally, the numerical scheme is tested on an analytical benchmark showing the accuracy of the method.

Note that the efficiency of such an algorithm had already been tested in the particular case of semi-Markov processes in [1].

2 The Theoretical Results

The following theorem characterizes the distribution $\pi(t)(i, dx)$ as the single solution of a set of integro-differential equations (the Chapman-Kolmogorov equations). Its proof may be found in [2].

Theorem 1: *Let $\pi_0(i, dx)$ be the initial distribution of the process (I_t , X_t). Then, under some technical assumptions, the distribution $\pi(t)(i, dx)$ is the* ***single*** *solution continuous in t of the following equations:*

$$\sum_{i\in E}\int_{\mathbb{R}^d} h(i,x)\,\pi(t)(i,dx) - \sum_{i\in E}\int_{\mathbb{R}^d} h(i,x)\,\pi_0(i,dx)$$
$$= \int_0^t \sum_{i\in E}\int_{\mathbb{R}^d} \left[\begin{array}{l} \sum_{j\in E} a(i,j,x)\left(\int_{\mathbb{R}^d} h(j,y)\mu(i,j,x)(dy) - h(i,x)\right) \\ +\mathbf{v}(i,x)\cdot\nabla h(i,x) \end{array} \right] \pi(s)(i,dx)\,ds$$

for all h continuously differentiable with a compact support.

We derive the following numerical scheme: for $\Delta t > 0$, the time-space $\mathbb{R}^+$ is partitioned into $\mathbb{R}_+ = \bigcup_{n\in\mathbb{N}}[n\Delta t, (n+1)\Delta t[$. Similarly, the space-state $\mathbb{R}^d$ is partitioned into what is called an admissible mesh M, namely a partition of $\mathbb{R}^d$ $(K)_{K\in M}$ such that $m_K \equiv \int_K dx > 0$ for all $K \in M$ and $h_M = sup_{K\in M}$ diam$(K) < \infty$.

For each $\Delta t > 0$ and each admissible mesh M, we construct a distribution $P_t(i, x)dx$ such that $P_t(i, x) \equiv p_n(i, K)$ is constant for all (t, x) such that $n\,\Delta t \le t < (n+1)\,\Delta t$ and such that $x \in K$, where $K \in M$.

For $n = 0$, we take $p_0(i,K)$ such that $m_K p_0(i,K) = \int_K \pi_0(i,dx)$, where $\pi_0(i, dx)$ is the initial distribution of the process (I_t , X_t).

The sequence $(p_n(i,K))_{n \in \mathbb{N}}$ is then recursively constructed through the following:

$$m_K p_{n+1}(i,K) = m_K \frac{U(i,n,K,\Delta t)}{\Delta t\, B(i,K) + m_K} + \Delta t \sum_{j \in E} \sum_{L \in M} A(i,j,K,L) \frac{U(j,n,L,\Delta t)}{\Delta t\, B(j,L) + m_L}$$

with

$$A(i,j,K,L) = \int_L a(j,i,y) \int_K \mu(j,i,y)(dx)dy$$

$$B(i,K) = \sum_{j \in E} \int_K a(i,j,x)dx$$

$$U(i,n,K,\Delta t) = \int_K P_n(i, g(i,x,-\Delta t)) J(i,x,-\Delta t)dx$$

where $P_n(i,x) = p_n(i,K)$ for all $x \in K$,

$g(i,x_0,t)$ is the single solution of $\frac{dx}{dt} = v(i,x)$ such that $g(i,x_0,0) = x_0$,

and $J(i,x_0,t)$ is the jacobian associated to the change of variable $x = g(i,x_0,t)$

It may be noted that this scheme is conservative, in the sense that $\sum_{i \in E} \sum_{K \in M} m_K p_n(i,K) = 1$ for all $n \in \mathbb{N}$. This allows us to check the limitation of the domain when implementing the scheme.

Theorem 2: *Under some technical assumptions, the distribution $P_t(i, x)\, dx$ associated to $\Delta t > 0$ and an admissible mesh M weakly converges towards the distribution $\pi(t)(i, dx)$ of the process (I_t , X_t) when Δt, h_M and $h_M / \Delta t$ tend to 0.*

The proof of this result may be found in [3]. The last assumption ($h_M / \Delta t$ tends to 0) is not always needed and Theorem 2 is true without such an assumption in several important cases (such as interacting semi-Markov processes for instance or for the analytical benchmark of next section).

3 An analytical benchmark

The numerical scheme presented in the previous section has been tested on different simple examples. Due to lack of room, we have chosen to focus here on one single analytical benchmark, which allows us to test different features of our scheme. This benchmark is taken from the PhD thesis of P.E. Labeau [4].

The studied benchmark is the following: a system is considered with three possible "physical" states: E = {1, 2, 3}. The system changes of physical state according to the following scheme: $1 \rightarrow 2 \rightarrow 3$ where $a(1,2, x) = \lambda_1$ and $a(2,3, x) =$

λ_2 are constant.

The "environmental" conditions X_t take their values in $\mathbb{R}^2$ and are assumed to be continuous, namely such that $\mu(i, j, (x,y))(dx',dy') = \delta_{(x,y)}(dx', dy')$ (for all $i, j, (x,y)$).

Beside, $X_t = (x_t , y_t)$ fulfils:

$$\begin{cases} \dfrac{dx_t}{dt} = a_i x_t \\ \dfrac{dy_t}{dt} = b_i x_t + c_i y_t \end{cases} \quad on \ \{I_t = i\} \ for \ \ i = 1,2,3$$

where $a_1 > a_2 > a_3 > c_3, \ a_1 \neq c_1, a_2 \neq c_2 , x_0 \neq 0$

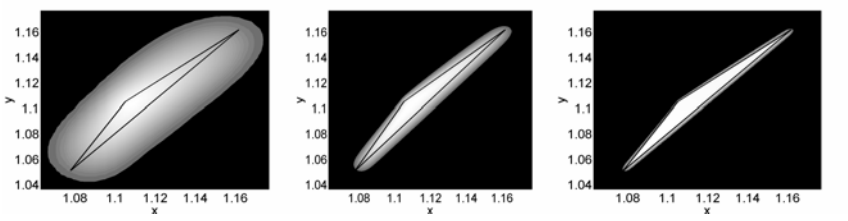
and $\dfrac{y_0}{x_0} = \dfrac{b_1}{a_1 - c_1} = \dfrac{b_2}{a_2 - c_2} > \dfrac{b_3}{a_3 - c_3}$.

Under such conditions, one may check that $\pi(t)(1, dx, dy)$ is supported by a point of $\mathbb{R}^2$ (it is a Dirac measure), that $\pi(t)(2, dx, dy)$ is supported by a line of $\mathbb{R}^2$ and that $\pi(t)(3, dx, dy)$ admit a density $\pi(t)(3, x, y)$ towards Lebesgue measure. Analytical expressions are available for all of those distributions and may be found in [4]. We here focus on $\pi(t)(3, x, y)$, which is the most interesting case.

Following [4], we take the following numerical values:

$a_1 = 1.5$	$a_2 = 1$	$a_3 = 0.75$	$b_1 = 1$	$b_2 = 0.75$	$b_3 = 0$
$c_1 = 0.5$	$c_2 = 0.25$	$c_3 = 0.5$	$x_0 = y_0 = 1$	$\lambda_1 = 0.5$	$\lambda_2 = 0.3$

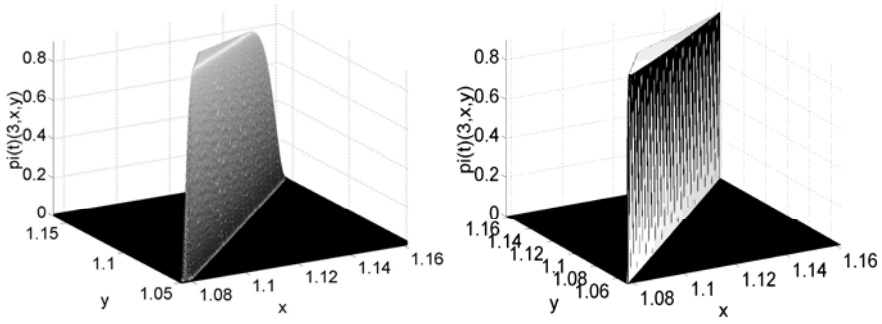
As for the set up of our numerical scheme, we take here the most simple mesh associated to $\mathbb{R}^2$ and we set $\mathbb{R}^2 = \bigcup_{m,n\in\mathbb{Z}} [mh,(m+1)h[\times[nh,(n+1)h[$ with $h = \Delta t$. Also, to program our scheme, we have to restrict ourselves to a domain in which the support of $\pi(t)(3, x, y)$ is expected to be included. Beside, the computing time is all the more reduced that this domain is small. Due to that, we perform consecutive executions of our program: we first take a rough step and a large domain. We then reduce the step of the mesh and select the area for the next execution. (At each step, approximated conservativeness of the mass is checked). Such successive restrictions of the domain are illustrated in Figures 1, 2 and 3 where the successive domains are displayed for $h = 5\times10^{-3}$, $h = 3{,}13\times10^{-4}$ and $h = 3{,}9\times10^{-5}$. The black triangle stands for the real support of $\pi(t)(3, x, y)$.



Figures 1, 2 & 3. The support of the approximate solution of $\pi(t)(3, x, y)$ for $h = 5\times10^{-3}$,

$h = 3{,}13\times10^{-4}$ and $h = 3{,}9\times10^{-5}$ and the exact support of $\pi(t)(3, x, y)$ (black triangle), $t = 0.1$

We now compare the full density $\pi(t)(i, x, y)$ as provided by our numerical scheme for the last step $h = 3{,}9\times10^{-5}$ (Figure 4) with the exact density (Figure 5). We also compare it with the method which gives the best results in [4] (Figure 6), where P.E. Labeau builds the full distribution by Monte-Carlo simulation step by step, taking into account both of the marginal distributions and of the support of the full distribution (cf [4] Chapter 4).



Figures 4 & 5. Approximate solution for $\pi(t)(3, x, y)$ with our numerical scheme for $h = 3{,}9\times10^{-5}$ (left) and exact distribution (right) for $t = 0.1$

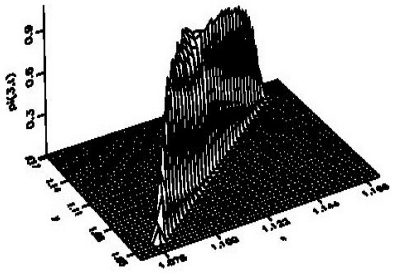


Figure 6. Approximate solution for $\pi(t)(3, x, y)$ with Monte-Carlo simulation step by step, PE Labeau [4] page 153 for $t = 0.1$

For $h = 3{,}9\times10^{-5}$, we can see that the numerical scheme provides us with quite good an approximation for $\pi(t)(3, x, y)$ which seems to be still better than the results obtained by PE Labeau in [4].

In order to complete the comparison with the method proposed in [4], we now plot the marginal densities $\pi(t)(3, x)$ and $\pi(t)(3, y)$, according to the analytical results (plain) and according to our numerical scheme (dashed) in Figures 7-8. The dashed plots are nearly indistinguishable from the plain plots. We also give the results

obtained by PE Labeau in [4] in Figures 9-10, which are very good too, but however a little more rugged than ours, as for the full distribution.

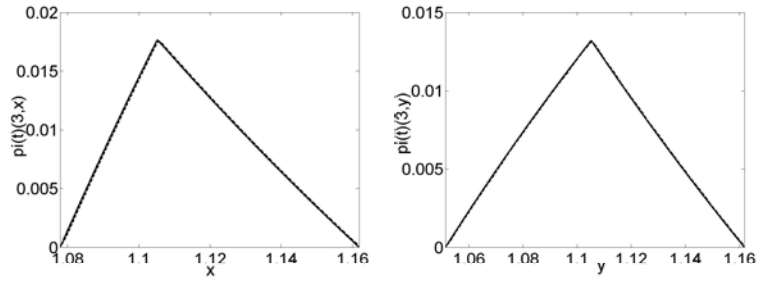


Figure 7 & 8. Exact marginal distributions $\pi(t)(3, x)$ and $\pi(t)(3, y)$ for $t = 0.1$ (plain) and approximate solutions with our numerical scheme for $h = 3{,}9\times10^{-5}$ (dashed)

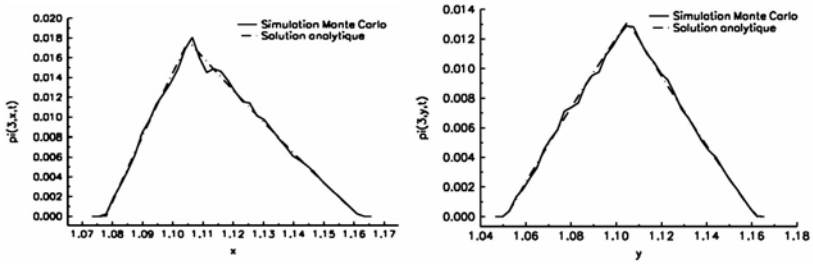


Figure 9 & 10. Exact marginal distributions $\pi(t)(3, x)$ and $\pi(t)(3, y)$ for $t = 0.1$ (plain) and approximate solutions with Monte-Carlo simulations (dashed)

As a conclusion, the results provided by the numerical scheme on this small analytical benchmark (and on others) are very good. The computing time is about three quarters of an hour on a modern PC to get Figure 4. It then seems to be a good alternative to Monte-Carlo simulation, at least when the dimension of the environmental condition is small (a few units).

Acknowledgement: Figures 6, 9 and 10 from [4] have been included here with the authorization of P. E. Labeau and the authors warmly thank him for that.

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