Sensitivity Estimates in Dynamic Reliability

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Abstract

The aim of this paper is to study and to compute first-order derivatives with respect to some parameter p, for some functionals of piecewise deterministic Markov processes (PDMP), in view of sensibility analysis in dynamic reliability. Such functionals are mean values of some function of the process, cumulated on some finite interval $[0, t]$, and their asymptotic value per unit time.

1   Introduction

In dynamic reliability, the time-evolution of a system is described by a piecewise deterministic Markov process (PDMP) $(I_t, X_t)_{t \geq 0}$ (see Davis 1993, Cocozza-Thivent & co 2006-1). The first component I_t is discrete, with values in a finite state space E. Typically, it indicates the state (up/down) for each component of the system at time t. The second component X_t, with values in $V \subset \mathbb{R}^d$, stands for environmental conditions, such as temperature, pressure, and so on. Both components of the process interact in each other: the transition rate for a jump of the discrete part I_t depends on the value of X_t just before the jump; between jumps of I_t, the evolution of X_t is deterministic with paths depending on the fixed discrete state of the system; by jump of I_t, the component X_t jumps to a random value which depends on the discrete states just before and after the jump. Under technical assumptions, $(I_t, X_t)_{t \geq 0}$ is a Markov process with general state space $E \times V$ (see Davis 1993, Cocozza-Thivent & co 2006-1).

We study quantities of the shape

$$R_{\rho_0}(t) = \mathbb{E}_{\rho_0} \left(\int_0^t h(I_s, X_s) \, ds \right)$$

where ρ_0 is the initial distribution of the process and h is some bounded measurable function.

We assume that the jump rates for I_t and the function h depend on some parameter p. The quantities of interest then are the first-order derivatives of $R_{\rho_0}(t)$ and $\lim_{t \to +\infty} R_{\rho_0}(t)/t$ with respect to p, which may help to rank input data according to their relative importance. This kind of sensitivity analysis was studied by Gandini (1990) and by Cao and Chen (1997) for Markov jump processes. In this paper we present extensions of their results to PDMP.

The model is presented in Section 2, as well as first results on differentiability of $R_{\rho_0}(t)$ with respect to p. In Section 3, we introduce importance functions, which are the main tool of the study. We derive an analytical expression for the derivative of $R_{\rho_0}(t)$ with respect to p. The asymptotic behaviour of the derivative of $R_{\rho_0}(t)/t$ with respect to p is studied in Section 4. We finally provide examples in Sections 5 and 6.

2   The model

The evolution of the process $(I_t, X_t)_{t \geq 0}$ is characterized by:

- the transition rate $a(i, j, x)$ from i to j when the environmental variable is equal to x; the function $a : E \times E \times V \to \mathbb{R}_+$ is assumed to be bounded, with $x \longmapsto a(i, j, x)$ continuous for all $i, j \in E$.

- the probability measure $\mu_{(i,j,x)}(dy)$ which controls jump of $(X_t)_{t\geq 0}$ by jumps of $(I_t)_{t\geq 0}$: given that $(I_{t^-}, X_{t^-}) = (i,x)$ just before a jump towards j at time t, the random variable X_t is then distributed according to $\mu_{(i,j,x)}(dy)$. We assume that for all $i,j \in E$ and for all function $\psi : V \to \mathbb{R}$ continuous and bounded, the function $x \longmapsto \mu_{(i,j,x)}\psi := \int \psi(y)\mu_{(i,j,x)}(dy)$ is continuous.

- the velocity $\mathbf{v}(i,x)$ of the environmental variable between two jumps when the discrete part is equal to i; given that $I_t = i$, X_t follows a deterministic trajectory which is solution of $\frac{dy}{dt} = \mathbf{v}(i,y)$; the function $\mathbf{v} : E \times V \to \mathbb{R}^d$ is assumed to be such that $x \longmapsto \mathbf{v}(i,x)$ is locally Lipschitz continuous and sub-linear for all $i \in E$ (there are some $V_1 > 0$ and $V_2 > 0$ such that $\forall i \in E, \forall x \in \mathbb{R}^d, \|\mathbf{v}(i,x)\| \leq V_1\|x\| + V_2$). These assumptions guarantee the existence and uniqueness of the solution to the differential equations fulfilled by the environmental component; we denote by $g(i,x,t)$ the single solution such that $g(i,x,0) = x$.

We assume that the jump rates $a(i,j,x)$ and the measurable bounded function h depend on some parameter p, where p belongs to an open set $O \subset \mathbb{R}$ or $\mathbb{R}^k$. We add exponent $^{(p)}$ to each quantity depending on p, such as $h^{(p)}$ or $R_{\rho_0}^{(p)}(t)$.

We denote by $\rho_t^{(p)}(j,dy)$ the distribution of the process $(I_t, X_t)_{t\geq 0}$ at time t with initial distribution ρ_0 (independent on p) and by $P_t^{(p)}(i,x,j,dy)$ the transition probability distribution of $(I_t, X_t)_{t\geq 0}$. We then have:

$$R_{\rho_0}^{(p)}(t) = \int_0^t \rho_s^{(p)} h^{(p)} \ ds = \sum_{i\in E} \int_V \left(\int_0^t h^{(p)}(i,x) \ ds \right) \rho_s^{(p)}(i,dx)$$

$$= \int_0^t \rho_0 P_s^{(p)} h^{(p)} \ ds = \sum_{i\in E} \int_V \left(\int_0^t \left(P_s^{(p)} h^{(p)} \right)(i,x) \ ds \right) \rho_0(i,dx)$$

Let us recall an expression for the transition probability distribution (see Cocozza-Thivent & co 2006-1):

Proposition 1 *The transition probability distribution of $(I_t, X_t)_{t\geq 0}$ is given by:*

$$\left(P_t^{(p)} f \right)(i,x) = \sum_{n=0}^{+\infty} \left(T_t^{(n)} f \right)(i,x)$$

for any function f bounded and measurable, with

$$\left(T_t^{(0)} f \right)(i,x) = f(i, g(i,x,t)) e^{-\int_0^t b^{(p)}(i,g(i,x,s))ds}$$

and for $n \geq 1$:

$$\left(T_t^{(n)} f \right)(i,x) = \sum_{j\neq i} \int_0^t c_i^{(p)}(u,x) \ a^{(p)}(i,j,g(i,x,u)) \left(\int_V \left(T_{t-u}^{(n-1)} f \right)(j,y) \, \mu_{(i,j,g(i,x,u))}(dy) \right) du$$

where $b^{(p)}(i,x) = \sum_{j\neq i} a^{(p)}(i,j,x)$ and $c_i^{(p)}(t,x) = e^{-\int_0^t b^{(p)}(i,g(i,x,s))ds}$.

In order to calculate derivatives of the functional $R_{\rho_0}^{(p)}$, we must give a sense to derivatives of the transition probability distributions. With that aim, we shall need the following assumption: the function $p \longmapsto a^{(p)}(i,j,x)$ is differentiable and, for all $p_0 \in O$, its derivative with respect to p is uniformly bounded for all $(i,x,p) \in E \times V \times N(p_0)$, where $N(p_0)$ is some neighbourhood of p_0 (**assumption $\mathcal{H}_1$**). We get the following result:

Proposition 2 *Under assumption $\mathcal{H}_1$, for all $i, j \in E$, all $x \in V$ and all $s \in [0, t]$, there exists a unique signed measure $\frac{\partial P_s^{(p)}}{\partial p}(i, x, j, dy)$ on V which is such that*

$$\frac{\partial}{\partial p}\left(\sum_{j \in E} \int_V f(j, y, s) P_s^{(p)}(i, x, j, dy)\right) = \sum_{j \in E} \int_{V_j} f(j, y, s) \frac{\partial P_s^{(p)}}{\partial p}(i, x, j, dy)$$

for all bounded measurable function f (independent of p). Moreover, we have:

$$\sup_{i, j \in E, x \in V, s \in [0, t]} \left| \frac{\partial P_s^{(p)}}{\partial p}(i, x, j, dx) \right| (V) < +\infty$$

Sketch of proof. Under $\mathcal{H}_1$, the expression of the transition probability distribution in Proposition 1 is differentiable with respect to p. The operator $f \to \left(\frac{\partial P_s^{(p)}}{\partial p} f\right)(i, x)$ is a linear operator and can be considered as a signed measure. ∎

We easily derive the following theorem:

Theorem 3 *We assume that $\mathcal{H}_1$ is true, that the function $p \longmapsto h^{(p)}(i, x)$ is differentiable and that, for all $p_0 \in O$, its derivative with respect to p is uniformly bounded for all $(i, x, p) \in E \times V \times N(p_0)$, where $N(p_0)$ is some neighbourhood of p_0 (**assumptions $\mathcal{H}_2$**). Then, the function $p \longmapsto R_{\rho_0}^{(p)}(s)$ is differentiable with respect to p and*

$$\frac{\partial}{\partial p}\left(R_{\rho_0}^{(p)}(s)\right) = \int_0^s \sum_{i \in E} \sum_{j \in E} \int_V \int_V h^{(p)}(j, y) \frac{\partial P_u^{(p)}}{\partial p}(i, x, j, dy) \rho_0(i, dx)\, du$$

$$+ \int_0^s \sum_{i \in E} \sum_{j \in E} \int_V \int_V \frac{\partial}{\partial p} h^{(p)}(j, y) P_u^{(p)}(i, x, j, dy) \rho_0(i, dx)\, du$$

$$= \int_0^s \sum_{j \in E} \int_V h^{(p)}(j, y) \frac{\partial \rho_u^{(p)}}{\partial p}(j, dy)\, du + \int_0^s \sum_{j \in E} \int_V \frac{\partial h^{(p)}}{\partial p}(j, y)\, \rho_u^{(p)}(j, dy)\, du \qquad (1)$$

where we set:

$$\frac{\partial \rho_s^{(p)}}{\partial p}(j, dy) := \sum_{i \in E} \int_V \frac{\partial P_s^{(p)}}{\partial p}(i, x, j, dy)\, \rho_0(i, dx)$$

Our purpose is to compute this derivative. We can compute the marginal distribution $\rho_u^{(p)}(j, dy)$ of the process by Monte-Carlo simulations or by the finite volume method from Cocozza-Thivent & co 2006-2 (at least when d and the number of discrete states are small). However, we do not know how to compute directly the derivatives of the marginal distribution which appear in the above expression. We now transform this expression in order to make it easier to compute.

3 Importance function

In this section we shall use the infinitesimal generator of the process:

Definition 4 *Let $\mathcal{D}_{H_0}$ be the set of functions $\varphi(i, x)$ from $E \times V$ to $\mathbb{R}$ such that for all $i \in E$ the function $x \longmapsto \varphi(i, x)$ is bounded and continuously differentiable and the function $x \longmapsto \mathbf{v}(i, x) \cdot \nabla \varphi(i, x)$ is bounded and continuous. For $\varphi \in \mathcal{D}_{H_0}$, we define*

$$H_0^{(p)} \varphi(i, x) = \sum_{j \in E} a^{(p)}(i, j, x)\left(\mu_{(i, j, x)} \varphi(j, \cdot)\right) + \mathbf{v}(i, x) \cdot \nabla \varphi(i, x)$$

with $a^{(p)}(i, i, x) = -\sum_{j \neq i} a^{(p)}(i, j, x)$ and $\mu_{(i, i, x)}(dy) = \delta_x(dy)$, where δ_x is the Dirac measure at x.

Let $\mathcal{D}_H$ be the set of functions $\varphi(i, x, s)$ from $E \times V \times \mathbb{R}$ to $\mathbb{R}$ such that for all $i \in E$ and $s \in \mathbb{R}_+$ the function $x \longmapsto \varphi(i, x, s)$ is bounded and continuously differentiable and the function $x \longmapsto \frac{\partial}{\partial s}\varphi(i, x, s) + \mathbf{v}(i, x) \cdot \nabla \varphi(i, x)$ is bounded and continuous. For $\varphi \in \mathcal{D}_H$, we define

$$H^{(p)}\varphi(i, x, s) = \sum_j a^{(p)}(i, j, x)\left(\mu_{(i,j,x)}\varphi(j, \cdot, s)\right) + \frac{\partial \varphi}{\partial s}(i, x, s) + \mathbf{v}(i, x) \cdot \nabla \varphi(i, x, s) \tag{2}$$

We then have: $P_s^{(p)}\varphi = \varphi + \int_0^s H_0^{(p)}\left(P_u^{(p)}\varphi\right)\,du$ for all $\varphi \in \mathcal{D}_{H_0}$ and $P_s^{(p)}\varphi(\cdot, \cdot, s) = \varphi(\cdot, \cdot, 0) + \int_0^s H^{(p)}\left(P_u^{(p)}\varphi\right)(\cdot, \cdot, u)\,du$ for all $\varphi \in \mathcal{D}_H$ (Chapman-Kolmogorov equations).

We are about to define the importance functions:

Definition 5 *We say that a function $\varphi_t^{(p)} \in \mathcal{D}_H$ is the importance function associated to the function $h^{(p)}$ and t if:*

- *$\varphi_t^{(p)}$ is solution of the differential equation $H^{(p)}\varphi_t^{(p)}(i, x, s) = h^{(p)}(i, x)$ for all $s \in [0, t[$,*
- *$\varphi_t^{(p)}(i, x, t) = 0$ for all (i, x) in $E \times V$.*

Such an importance function may be proved to be uniquely associated to $\left(h^{(p)}, t\right)$, due to the Cauchy-Lipschitz theorem and using a similar method as in Cocozza-Thivent & co (2006-1).

In examples from Sections 5 and 6, the importance functions will be computed numerically. However, an analytical form is available, which is also useful for the asymptotic study:

Lemma 6 *Let us assume that the function $x \longmapsto a^{(p)}(i, j, x)$ is continuously differentiable for all $i, j \in E$, all $x \in V$ and all $p \in O$, and that the function $\mathbf{v}$ is bounded (**assumptions** $\mathcal{H}_3$), the importance function associated to $\left(h^{(p)}, t\right)$ is then given by:*

$$\varphi_t^{(p)}(i, x, s) = \begin{cases} -\int_0^{t-s}\left(P_u^{(p)}h^{(p)}\right)(i, x)\,du & \text{if } 0 \le s \le t \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Sketch of proof. It is clear that $\varphi_t^{(p)}(i, x, t) = 0$ and under $\mathcal{H}_3$, one may check that the function $\varphi_t^{(p)}(i, x, s)$ is in $\mathcal{D}_H$. Beside, for $0 \le s \le t$, we have:

$$\left(H^{(p)}\varphi_t^{(p)}\right)(.,.,s) = -H^{(p)}\int_0^{t-s}P_u^{(p)}h^{(p)}\,du = -H_0^{(p)}\left(\int_0^{t-s}P_u^{(p)}h^{(p)}\,du\right) - \frac{\partial}{\partial s}\left(\int_0^{t-s}P_u^{(p)}h^{(p)}\,du\right)$$

$$= -\int_0^{t-s}H_0^{(p)}\left(P_u^{(p)}h^{(p)}\right)\,du + P_{t-s}^{(p)}h^{(p)}$$

$$= h^{(p)}$$

due to the Chapman-Kolmogorov equation, which ends the proof. ∎

We now derive a new expression for $\frac{\partial R_{\rho_0}^{(p)}}{\partial p}(t)$:

Theorem 7 *Under assumptions $\mathcal{H}_{1-3}$ (namely $\mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3$), we have:*

$$\frac{\partial R_{\rho_0}^{(p)}}{\partial p}(t) = \int_0^t \rho_s^{(p)}\frac{\partial h^{(p)}}{\partial p}\,ds + \int_0^t \rho_s^{(p)}\frac{\partial H^{(p)}}{\partial p}\left(\int_0^{t-s}P_u^{(p)}h^{(p)}\,du\right)ds \tag{4}$$

where we set:

$$\frac{\partial H^{(p)}}{\partial p}\varphi(i, x, s) := \sum_{j \in E}\frac{\partial a^{(p)}}{\partial p}(i, j, x)\left(\mu_{(i,j,x)}\varphi(j, \cdot, s)\right)$$

for all $\varphi \in \mathcal{D}_H$, all $(i, x, s) \in E \times V \times \mathbb{R}_+$.

Proof. Starting from (1), we have to compute $\int_0^t \frac{\partial \rho_s^{(p)}}{\partial p} h^{(p)}(\cdot, \cdot, s) \ ds$. We first know from the Chapman-Kolmogorov equation applied to $\varphi_t^{(p)}$ that:

$$\int_0^t \rho_s^{(p)} H^{(p)} \varphi_t^{(p)}(.,.,s) \ ds = \rho_t^{(p)} \varphi_t^{(p)}(.,.,t) - \rho_0 \varphi_t^{(p)}(.,.,0) = -\rho_0 \varphi_t^{(p)}(.,.,0)$$

due to $\varphi_t^{(p)}(\cdot, \cdot, t) = 0$. By differentiating this expression with respect to p, we derive:

$$\int_0^t \frac{\partial \rho_s^{(p)}}{\partial p} h^{(p)}(\cdot, \cdot, s) \ ds + \int_0^t \rho_s^{(p)} \frac{\partial H^{(p)}}{\partial p} \varphi_t^{(p)}(\cdot, \cdot, s) \ ds + \int_0^t \rho_s^{(p)} H^{(p)} \frac{\partial \varphi_t^{(p)}}{\partial p}(\cdot, \cdot, s) \ ds = -\rho_0 \frac{\partial \varphi_t^{(p)}}{\partial p}(\cdot, \cdot, 0)$$
(5)

Chapman-Kolmogorov equation applied to $\frac{\partial \varphi_t^{(p)}}{\partial p}$ gives:

$$\int_0^t \rho_s^{(p)} H^{(p)} \frac{\partial \varphi_t^{(p)}}{\partial p}(\cdot, \cdot, s) \ ds = \rho_t^{(p)} \frac{\partial \varphi_t^{(p)}}{\partial p}(\cdot, \cdot, t) - \rho_0 \frac{\partial \varphi_t^{(p)}}{\partial p}(\cdot, \cdot, 0) = -\rho_0 \frac{\partial \varphi_t^{(p)}}{\partial p}(\cdot, \cdot, 0)$$

We derive from (5):

$$\int_0^t \frac{\partial \rho_s^{(p)}}{\partial p} h^{(p)}(\cdot, \cdot) \ ds = -\int_0^t \rho_s^{(p)} \frac{\partial H^{(p)}}{\partial p} \varphi_t^{(p)}(\cdot, \cdot, s) \ ds$$

Whence the result, using (1) and substituting $\varphi_t^{(p)}$ with (3). ∎

Equation (4) actually is an extension of the results given by Gandini (1990) for jump Markov processes.

4   Asymptotic results

In all this section, we assume that the process $(I_t, X_t)_{t \geq 0}$ is positive Harris-recurrent with $\pi^{(p)}$ as unique stationary distribution. We first transform (4) in view of studying its asymptotic expression:

Lemma 8 *Under assumptions $\mathcal{H}_{1-3}$, we have:*

$$\frac{1}{t} \frac{\partial R_{\rho_0}^{(p)}}{\partial p}(t) = \frac{1}{t} \int_0^t \rho_s^{(p)} \frac{\partial h^{(p)}}{\partial p} \ ds + \frac{1}{t} \int_0^t \rho_s^{(p)} \frac{\partial H^{(p)}}{\partial p} \left(\int_0^{t-s} \left(P_u^{(p)} h^{(p)} - \pi^{(p)} h^{(p)} \right) \ du \right) ds \qquad (6)$$

Proof. The first term is clear. Beside, setting $\mathbf{1}$ the constant function equal to 1, we have: $\frac{\partial H^{(p)}}{\partial p} \mathbf{1} = 0$ since $H^{(p)} \mathbf{1} = 0$. As $\pi^{(p)} h^{(p)}$ is a constant (independent of (i, x)), we derive

$$\frac{\partial H^{(p)}}{\partial p} \pi^{(p)} h^{(p)} = \left(\pi^{(p)} h^{(p)} \right) \frac{\partial H^{(p)}}{\partial p} \mathbf{1} = 0$$

and consequently:

$$\frac{\partial H^{(p)}}{\partial p} \left(\int_0^{t-s} \pi^{(p)} h^{(p)} \ du \right) = (t - s) \frac{\partial H^{(p)}}{\partial p} \pi^{(p)} h^{(p)} = 0$$

Whence the result. ∎

We may now prove existence and provide an asymptotic expression for $\frac{1}{t} \frac{\partial R_{\rho_0}^{(p)}}{\partial p}(t)$, at least under the following additional assumptions: we assume that, for each $p \in O$, there exists a function $f^{(p)}$ such that $\int_0^{+\infty} f^{(p)}(u) \ du < +\infty$, $\int_0^{+\infty} u \ f^{(p)}(u) \ du < +\infty$, $\lim_{u \to +\infty} f^{(p)}(u) = 0$ and $\left| \left(P_u^{(p)} h^{(p)} \right)(i, x) - \pi^{(p)} h^{(p)} \right| \leq f^{(p)}(u)$ for all $(i, x) \in E \times V$, all $u \geq 0$ (**assumptions $\mathcal{H}_4$**).

Theorem 9 *Let us assume that $\mathcal{H}_{1-4}$ are true. Then:*

$$Uh^{(p)}(i, x) := \int_0^{+\infty} \left(\left(P_u^{(p)} h^{(p)} \right)(i, x) - \pi^{(p)} h^{(p)} \right) \ du$$

exists for all $(i, x) \in E \times V$ and

$$\lim_{t \to +\infty} \frac{1}{t} \frac{\partial R_{\rho_0}^{(p)}}{\partial p}(t) = \pi^{(p)} \frac{\partial h^{(p)}}{\partial p} + \pi^{(p)} \frac{\partial H_0^{(p)}}{\partial p} Uh^{(p)} \qquad (7)$$

Proof. The quantity $Uh^{(p)}(i,x)$ is clearly defined for all $(i,x) \in E \times V$. To derive (7) from (6), we use the fact that, due to positive Harris-recurrence of $(I_t, X_t)_{t \geq 0}$, we know:

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \rho_s^{(p)} \varphi^{(p)} \, ds = \pi^{(p)} \varphi^{(p)} \tag{8}$$

for each measurable and bounded $\varphi^{(p)}$ (see Asmussen (1987)). Under $\mathcal{H}_2$, the first term in right side of (6) consequently converges to the first term in (7). For the second term, setting $U_s := \int_s^{+\infty} \left(P_u^{(p)} h^{(p)} - \pi^{(p)} h^{(p)} \right) \, du$, we first have:

$$\left| \frac{\partial H^{(p)}}{\partial p} U_s(i,x) \right| \leq \sum_{j \in E} \left| \frac{\partial a^{(p)}}{\partial p}(i,j,x) \left(\mu_{(i,j,x)} U_s(j,\cdot) \right) \right|$$

As $|U_s| \leq \int_s^{+\infty} f^{(p)}(u) du$ due to $\mathcal{H}_4$, we also have $\left| \left(\mu_{(i,j,x)} U_s(j,\cdot) \right) \right| \leq \int_s^{+\infty} f^{(p)}(u) du$ for all (i,j,x,s). Using $\mathcal{H}_1$, we get existence of $K \geq 0$ such that:

$$\left| \frac{\partial H^{(p)}}{\partial p} U_s(i,x) \right| \leq K \int_s^{+\infty} f^{(p)}(u) du \tag{9}$$

for all (i,x,s). We derive:

$$\left| \frac{1}{t} \int_0^t \rho_s^{(p)} \frac{\partial H^{(p)}}{\partial p} U_{t-s} ds \right| \leq \frac{K}{t} \int_0^t \rho_s^{(p)} \int_{t-s}^{+\infty} f^{(p)}(u) du \, ds$$

$$= \frac{K}{t} \int_0^t \int_{t-s}^{+\infty} f^{(p)}(u) du \, ds$$

$$= \frac{K}{t} \int_0^{+\infty} u \, f^{(p)}(u) \, du$$

so that $\lim_{t \to +\infty} \frac{1}{t} \int_0^t \rho_s^{(p)} \frac{\partial H^{(p)}}{\partial p} U_{t-s} ds = 0$ due to $\mathcal{H}_4$. Beside, (9) shows that $\frac{\partial H^{(p)}}{\partial p} Uh^{(p)}$ is bounded due to $\mathcal{H}_4$ again. We derive from (8) that

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \rho_s^{(p)} \frac{\partial H^{(p)}}{\partial p} Uh^{(p)} ds = \pi^{(p)} \frac{\partial H^{(p)}}{\partial p} Uh^{(p)} = \pi^{(p)} \frac{\partial H_0^{(p)}}{\partial p} Uh^{(p)}$$

because $Uh^{(p)}$ is independent on time. Whence the result, using

$$\int_0^{t-s} \left(P_u^{(p)} h^{(p)} - \pi^{(p)} h^{(p)} \right) \, du = Uh^{(p)} - U_{t-s}$$

in (6) and letting $t \to +\infty$. $\blacksquare$

The previous theorem provides an extension of the results given by Cao and Chen (1997) for jump Markov processes.

The following proposition now gives a tool to compute the function Uh.

Proposition 10 *Let us assume $\mathcal{H}_{1-4}$ to be true. The function $Uh^{(p)}$ fulfills the differential equation:*

$$H_0^{(p)} Uh^{(p)}(i,x) = \pi^{(p)} h^{(p)} - h^{(p)}(i,x)$$

Sketch of proof. We have:

$$H_0^{(p)} Uh^{(p)}(i,x) = H_0^{(p)} \int_0^{+\infty} \left(\left(P_u^{(p)} h^{(p)} \right)(i,x) - \pi^{(p)} h^{(p)} \right) \, du$$

$$= \int_0^{+\infty} \left(H_0^{(p)} P_u^{(p)} h^{(p)}(i,x) - H_0^{(p)} \left(\pi^{(p)} h^{(p)} \right) \right) \, du$$

$$= \int_0^{+\infty} H_0^{(p)} P_u^{(p)} h^{(p)}(i,x) \, du$$

since $H_0^{(p)}\left(\pi^{(p)}h^{(p)}\right) = \left(\pi^{(p)}h^{(p)}\right)H_0^{(p)}\mathbf{1} = 0$. We derive:

$$
\begin{aligned}
H_0^{(p)}Uh^{(p)}(i,x) &= \lim_{t\to+\infty}\int_0^t\left(H_0^{(p)}P_u^{(p)}h^{(p)}(i,x)\right)\,du\\
&= \lim_{t\to+\infty}P_t^{(p)}h^{(p)}(i,x) - h^{(p)}(i,x) \quad \text{(Chapman-Kolmogorov equation)}\\
&= \pi^{(p)}h^{(p)} - h^{(p)}(i,x)
\end{aligned}
$$

due to $\mathcal{H}_4$. $\blacksquare$

We now look at two examples. In such examples, dependence on p (namely $^{(p)}$) is generally not specified any more, in order to get simpler notations.

5 A first example

A single component is considered, which is perfectly and instantaneously repaired at each failure. The time evolution of the component is described by the process $(X_t)_{t\geq 0}$ where X_t stands for the time elapsed at time t since the last instantaneous repair. (There is one single discrete state here so that the component I_t is not necessary). The failure rate for the component at time t is $\lambda(X_t)$ where $\lambda(\cdot)$ is some continuous non negative function. The process $(X_t)_{t\geq 0}$ is "renewed" after each repair so that $\mu(x)(dy) = \delta_0(dy)$ and the evolution of $(X_t)_{t\geq 0}$ between renewals is given by $g(x,t) = x + t$.

We are interested in the rate of renewals on $[0,t]$, namely in the quantity $Q(t)$ such that:

$$
Q(t) = \frac{R(t)}{t} = \frac{1}{t}\mathbb{E}_0\left(\int_0^t\lambda(X_s)\,ds\right) = \frac{1}{t}\int_0^t\left(\int_{\mathbb{R}_+}\lambda(x)\rho_s(dx)\right)ds
$$

where $R(t)$ is the renewal function associated to the underlying renewal process and ρ_s is the distribution of X_s given that $X_0 = 0$.

The function $\lambda(x)$ depends on some parameter p and we want to compute $\frac{\partial Q(t)}{\partial p}$. Using (4), we get:

$$
\frac{\partial Q(t)}{\partial p} = \frac{1}{t}\int_0^t\int_0^s\rho_s(dx)\frac{\partial\lambda}{\partial p}(x)\left(1 - \varphi_t(0,s) + \varphi_t(x,s)\right)\,ds
$$

where φ_t is solution of $\lambda(x)(\varphi_t(0,s) - \varphi_t(x,s)) + \frac{\partial}{\partial s}\varphi_t(x,s) + \frac{\partial}{\partial x}\varphi_t(x,s) = \lambda(x)$ for all $s \in [0,t[$ and $\varphi_t(x,t) = 0$ for all $x \in [0,t]$. No closed form is available for φ_t and for the numerical computation, this equation has been discretized and solved numerically.

As for the asymptotic quantities, assuming $\mathbb{E}(T_1) < +\infty$, where T_1 is the first renewal time, it is known that:

$$
Q(\infty) = \int_{\mathbb{R}_+}\lambda(x)\pi(dx) = \frac{1}{\mathbb{E}(T_1)}
$$

where π is the stationary distribution of $(X_t)_{t\geq 0}$. Beside, π has a density f_π with respect to Lebesgue measure with:

$$
f_\pi(x) = \frac{\mathbb{P}_0(T_1 > x)}{\mathbb{E}(T_1)} = \frac{e^{-\int_0^x\lambda(u)du}}{\mathbb{E}(T_1)} \tag{10}
$$

Now, we know from (7) that:

$$
\frac{\partial Q(\infty)}{\partial p} = \pi\frac{\partial\lambda}{\partial p}(1 + V\lambda(0) - V\lambda) \tag{11}
$$

where $V\lambda$ is solution of $\lambda(x)(V\lambda(0) - V\lambda(x)) + \frac{\partial}{\partial x}(V\lambda(x)) = \pi\lambda - \lambda(x) = Q(\infty) - \lambda(x)$.

Solving this equation and substituting in (11), we get:

$$
\begin{aligned}
\frac{\partial Q(\infty)}{\partial p} &= \int_0^{+\infty}f_\pi(x)\frac{\partial\lambda}{\partial p}(x)\left(1 - \int_0^x(Q(\infty) - \lambda(v))e^{\int_v^x\lambda(u)du}dv\right)dx\\
&= \frac{1}{\mathbb{E}(T_1)}\int_0^{+\infty}\frac{\partial\lambda}{\partial p}(x)\left(1 - Q(\infty)\int_0^x e^{-\int_0^v\lambda(u)du}dv\right)dx
\end{aligned}
$$

using (10).

Taking $\lambda\left(t\right) = \alpha\beta t^{\beta-1}$ and $\left(\alpha,\beta\right) = \left(10^{-5},4\right)$, we are now able to compute $\frac{\partial Q(t)}{\partial \alpha}$ and $\frac{\partial Q(t)}{\partial \beta}$ for $t \le \infty$. In order to validate our results, we also compute such quantities by finite differences using:

$$\frac{\partial Q\left(t\right)}{\partial p} \simeq \frac{1}{\varepsilon}\left(Q^{(p+\varepsilon)}\left(t\right) - Q^{(p)}\left(t\right)\right)$$

for small ε and $t \le \infty$. For the asymptotic results, we use $Q\left(\infty\right) = \frac{1}{\mathbb{E}(T_1)}$ to compute such a derivative. For the transitory results, we use the algorithm from Mercier (2007) which provides the renewal function $R\left(t\right)$ and hence $Q\left(t\right) = \frac{R(t)}{t}$. The results are gathered in Table 1 for the asymptotic derivatives.

Table 1: $\frac{\partial Q(\infty)}{\partial \alpha}$ and $\frac{\partial Q(\infty)}{\partial \beta}$ by finite differences (FD) and the present method (MR)

	FD				MR
ε	10^{-4}	10^{-6}	10^{-8}	10^{-10}	
$\frac{\partial Q(\infty)}{\partial \alpha}$	5.1×10^2	1.496×10^3	1.5504×10^3	1.5510×10^3	1.5509×10^3
ε	10^{-4}	10^{-6}	10^{-8}	10^{-10}	
$\frac{\partial Q(\infty)}{\partial \beta}$	4.3761×10^{-2}	4.3760×10^{-2}	4.3760×10^{-2}	4.3760×10^{-2}	4.3755×10^{-2}

The results are very stable for $\frac{\partial Q(\infty)}{\partial \beta}$ by FD choosing different values for ε and FD give very similar results as MR. The approximation for $\frac{\partial Q(\infty)}{\partial \alpha}$ by FD requires smaller ε to give similar results as MR.

We now plot the transitory results in Figures 1 and 2 for $t \in [0,50]$.
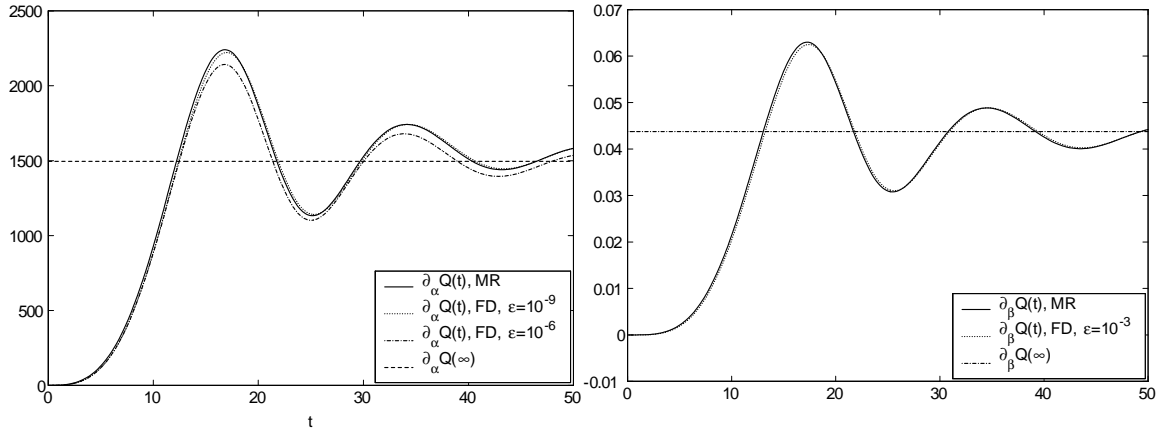
Figures 1 and 2: $\frac{\partial Q(t)}{\partial \alpha}$ and $\frac{\partial Q(t)}{\partial \beta}$ by FD and MR

In Figure 2, the results are very similar for $\frac{\partial Q(t)}{\partial \beta}$ by FD and MR even for ε not that small (here $\varepsilon = 10^{-3}$) whereas Figure 1 shows that FD requires much smaller ε ($\varepsilon \le 10^{-8}$) for $\frac{\partial Q(t)}{\partial \alpha}$ to provide similar results as MR.

6 A second example

6.1 Presentation - Theoretical results

A tank is considered, which may be filled in or emptied out using a pump. This pump may be in two different states: "in" (state 0) or "out" (state 1). The level of liquid in the tank goes from 0 up to R. The state of the system "tank-pump" at time t is (I_t, X_t) where I_t is the discrete state of the pump ($I_t \in \{0,1\}$) and X_t is the continuous level in the tank ($X_t \in [0,R]$). The transition rate from state 0

(resp. 1) to state 1 (resp. 0) at time t is $\lambda_0(X_t)$ (resp. $\lambda_1(X_t)$). The speed of variation for the liquid level in state 0 is $\mathbf{v}_0(x) = r_0(x)$ with $r_0(x) > 0$ for all $x \in [0, R[$ and $r_0(R) = 0$: the level increases in state 0 up to reaching R, where it remains constant. Similarly, the speed in state 1 is $\mathbf{v}_1(x) = -r_1(x)$ with $r_1(x) > 0$ for all $x \in]0, R]$ and $r_1(0) = 0$: the level of liquid decreases in state 1 until reaching 0, where it remains constant. Also, the level in the tank is continuous so that $\mu(i, 1-i, x)(dy) = \delta_x(dy)$ for $i \in \{0, 1\}$, all $x \in [0, R]$. The functions r_i and λ_i are assumed to be continuous, with λ_i bounded, which ensures an almost sure finite number of jumps on $[0, t]$ (all $t \geq 0$).

Such an example is very similar to that from Boxma & co (2005). The main difference is that we here assume X_t to remain bounded ($X_t \in [0, R]$) whereas X_t takes its values in $\mathbb{R}_+$ in the quoted paper.

In order to study asymptotic quantities, we assume conditions which ensures the process $(I_t, X_t)_{t \geq 0}$ to be φ-irreducible, in the sense of Down, Meyn and Tweedie (1996). Such conditions for irreducibility are very similar to those by Boxma & co (2005): we first take $\lambda_1(0) > 0$ and $\lambda_0(R) > 0$ which prevents the system from being stuck in states $(1, 0)$ and $(0, R)$, respectively. Setting $t_{x \to y}^{(i)}$ for the deterministic time to go from x up to y following the curve $(g(i, x, t))_{t \in \mathbb{R}}$ (all $x, y \in [0, R]$), we also assume that:

$$\text{if } \int_x^R \frac{1}{r_0(u)} du = t_{x \to R}^{(0)} = +\infty, \text{ then } \int_x^R \frac{\lambda_0(u)}{r_0(u)} du = +\infty \text{ for some (and hence all) } x \in [0, R[\quad (12)$$

$$\text{if } \int_0^y \frac{1}{r_1(u)} du = t_{y \to 0}^{(1)} = +\infty, \text{ then } \int_0^y \frac{\lambda_1(u)}{r_1(u)} du = +\infty \text{ for some (and hence all) } y \in]0, R] \quad (13)$$

The first condition ensures that if $t_{x \to R}^{(0)}$ is infinite, then the probability for the process $(X_t)_{t \geq 0}$ starting from x to reach R without any jump is zero, the same for $t_{y \to 0}^{(1)}$ and the second condition.

Due to the general assumptions, we also have:

$$\int_x^y \frac{1}{r_i(u)} du = t_{x \to y}^{(i)} < +\infty \text{ and } \int_x^y \frac{\lambda_i(u)}{r_i(u)} du = \int_0^{t_{x \to y}^{(i)}} \lambda_i(g(i, x, v)) dv < +\infty$$

for all $0 < x < y < R$. The second expression ensures that the probability for $(X_t)_{t \geq 0}$ starting from x to reach y without any jump is non zero (if x and y are correctly ordered according to the monotony of $t \longmapsto g(i, x, t)$). Such conditions are assumed in Boxma & co (2005) to ensure irreducibility but seem to be always true here.

To sum up, conditions for irreducibility here are: $\lambda_1(0) > 0$, $\lambda_0(R) > 0$ and $(12 - 13)$. Such conditions are referred to as **assumption** $\mathcal{H}_I$ in the following.

Proposition 11 *Under assumption* $\mathcal{H}_I$, *the process* $(I_t, X_t)_{t \geq 0}$ *is positive Harris recurrent with single invariant distribution* π *given by:*

$$\pi(i, dx) = f_i(x) dx$$

for $i = 0, 1$ *and*

$$f_0(x) = \frac{K_\pi}{v_0(x)} e^{-\int_{R/2}^x \left(\frac{\lambda_1(u)}{v_1(u)} + \frac{\lambda_0(u)}{v_0(u)} \right) du} = \frac{K_\pi}{r_0(x)} e^{\int_{R/2}^x \left(\frac{\lambda_1(u)}{r_1(u)} - \frac{\lambda_0(u)}{r_0(u)} \right) du} \quad (14)$$

$$f_1(x) = -\frac{K_\pi}{v_1(x)} e^{-\int_{R/2}^x \left(\frac{\lambda_1(u)}{v_1(u)} + \frac{\lambda_0(u)}{v_0(u)} \right) du} = \frac{K_\pi}{r_1(x)} e^{\int_{R/2}^x \left(\frac{\lambda_1(u)}{r_1(u)} - \frac{\lambda_0(u)}{r_0(u)} \right) du} \quad (15)$$

where $K_\pi > 0$ *is a normalization constant.*

Remark 12 *Though such results are very similar to some special case from Boxma & co (2005), we have better give here a quick proof due to a few differences in the results, such as some eventual masses for* π *at the bounds of the interval in the quoted paper.*

Sketch of proof. Under $(\mathcal{H}_I)$, one may first prove that the process $(I_t, X_t)_{t \geq 0}$ with values in $F := \{0, 1\} \times [0, R]$ is φ-irreducible for $\varphi = c_{\{0,1\}} \times \lambda$ where $c_{\{0,1\}}$ is the counting measure on $\{0, 1\}$ and λ is the Lebesgue measure on $[0, R]$. Beside, the process $(I_t, X_t)_{t \geq 0}$ is non-evanescent (due to values in a compact set) and it is a T-process (a proof may be found in Desgrouas (2007) in more general a context). The process is then Harris recurrent (Meyn-Tweedie 1993) and it admits a unique invariant measure π up to some multiplicative constant.

As the support of φ is the whole set F, the irreducibility measure $\varphi = c_{\{0,1\}} \times \lambda$ actually is maximal. As a consequence, π and $\varphi = c_{\{0,1\}} \times \lambda$ are mutually absolutely continuous (Down, Meyn and Tweedie 1996). We can then write:

$$\pi(i, dx) = f_i(x) \, dx$$

for some positive measurable function f_i. Beside, using the fact that $\pi(\cdot, dx)$ is such that $\pi H_0^{(p)} \varphi = 0$ for all $\varphi \in C_1([0, R])$, one easily find that

$$\lambda_{1-i}(x) f_{1-i}(x) - \lambda_i(x) f_i(x) - \frac{d}{dx}(v_i(x) f_i(x)) = 0$$

for $i = 0, 1$, so that (f_0, f_1) are of the shape $(14 - 15)$. Beside, it is easy to check that, for $i = 0, 1$: $\int_0^R f_i(x) \, dx < \infty$. We derive that π is a finite measure which can then be normalized in a single way in a probability measure. Consequently, (I_t, X_t) is a *positive* Harris recurrent process, which ends the proof. ∎

6.2 Quantities of interest

We are interested in two quantities: first, the proportion of time spent by the level in the tank between two fixed bounds a and b with $0 < a < b < R$ and we set:

$$Q_1(t) = \frac{1}{t} \mathbb{E}_{\rho_0}\left(\int_0^t \mathbf{1}_{\{a \leq X_s \leq b\}} ds\right) = \frac{1}{t} \sum_{i=0}^1 \int_0^t \int_a^b \rho_s(i, dx) \, ds = \frac{1}{t} \int_0^t \rho_s h_1 \, ds \qquad (16)$$

with $h_1(i, x) = \mathbf{1}_{[a,b]}(x)$.

The second quantity of interest is the mean number of times the pump is turned from state "in" (0) to state "out" (1) by unit time, namely:

$$Q_2(t) = \frac{1}{t} \mathbb{E}_{\rho_0}\left(\sum_{0 < s \leq t} \mathbf{1}_{\{I_{s-}=0 \text{ and } I_s=1\}}\right) = \frac{1}{t} \mathbb{E}_{\rho_0}\left(\int_0^t \lambda_0(X_s) \mathbf{1}_{\{I_s=0\}} ds\right)$$

$$= \frac{1}{t} \int_0^t \int_0^R \lambda_0(x) \rho_s(0, dx) \, ds = \frac{1}{t} \int_0^t \rho_s h_2 \, ds \qquad (17)$$

with $h_2(i, x) = \mathbf{1}_{\{i=0\}} \lambda_0(x)$.

For both quantities ($Q_1(t)$ and $Q_2(t)$), we want to study the influence of some parameter α_i on which depends $\lambda_i(x)$ but neither $\lambda_{1-i}(x)$, nor $v_0(x)$, $v_1(x)$, $\rho_0(\cdot, dx)$, R, a nor b. More precisely, we want to compute $\frac{\partial Q_{i_0}(t)}{\partial \alpha_{i_1}}$ and $\frac{\partial Q_{i_0}(\infty)}{\partial \alpha_{i_1}}$ for $i_0, i_1 \in \{0, 1\}$, where $Q_{i_0}(\infty) = \lim_{t \to +\infty} Q_{i_0}(t)$.

Setting $\varphi_t^{(i_0)}(i_1, x, s) = -\int_0^{t-s}(P_u h_{i_0})(i_1, x) \, du$ for the importance function associated to h_{i_0}, we first know from (4) that, for $i_1 \in \{0, 1\}$:

$$\frac{\partial Q_1(t)}{\partial \alpha_{i_1}} = \frac{1}{t} \int_0^R \int_0^t \rho_s(i_1, dx) \frac{\partial \lambda_{i_1}(x)}{\partial \alpha_{i_1}}\left(\varphi_t^{(1)}(i_1, x, s) - \varphi_t^{(1)}(1 - i_1, x, s)\right) ds \qquad (18)$$

and:

$$\frac{\partial Q_2(t)}{\partial \alpha_0} = \frac{1}{t} \int_0^R \int_0^t \rho_s(0, dx) \frac{\partial \lambda_0(x)}{\partial \alpha_0}\left(1 - \varphi_t^{(2)}(1, x, s) + \varphi_t^{(2)}(0, x, s)\right) ds \qquad (19)$$

$$\frac{\partial Q_2(t)}{\partial \alpha_1} = \frac{1}{t} \int_0^R \int_0^t \rho_s(1, dx) \frac{\partial \lambda_1(x)}{\partial \alpha_1}\left(\varphi_t^{(2)}(1, x, s) - \varphi_t^{(2)}(0, x, s)\right) \qquad (20)$$

As for the asymptotic derivatives, using (7), we get:

$$\frac{\partial Q_1(\infty)}{\partial \alpha_{i_1}} = \int_0^R \pi(i_1, dx) \frac{\partial \lambda_{i_1}(x)}{\partial \alpha_{i_1}} (Uh_1(1 - i_1, x) - Uh_1(i_1, x)) \tag{21}$$

for $i_1 \in \{0, 1\}$ and

$$\frac{\partial Q_2(\infty)}{\partial \alpha_0} = \int_0^R \pi(0, dx) \frac{\partial \lambda_0(x)}{\partial \alpha_0} (1 + Uh_2(1, x) - Uh_2(0, x)) \tag{22}$$

$$\frac{\partial Q_2(\infty)}{\partial \alpha_1} = \int_0^R \pi(1, dx) \frac{\partial \lambda_1(x)}{\partial \alpha_1} (Uh_2(0, x) - Uh_2(1, x)) \tag{23}$$

where, in such expressions, the function Uh_{i_0} is solution of

$$v_i(x) \frac{d}{dx} (Uh_{i_0}(i, x)) + \lambda_i(x) (Uh_{i_0}(1 - i, x) - Uh_{i_0}(i, x)) = Q_{i_0}(\infty) - h_{i_0}(i, x)$$

for $i = 0, 1$. One easily gets:

$$Uh_{i_0}(1, x) - Uh_{i_0}(0, x) = -\int_0^x \left(\frac{u_{i_0}(1, z)}{r_1(z)} + \frac{u_{i_0}(0, z)}{r_0(z)} \right) e^{\int_x^z \left(\frac{\lambda_1(y)}{r_1(y)} - \frac{\lambda_0(y)}{r_0(y)} \right) dy} < +\infty \tag{24}$$

and closed forms are now available for $Q_{i_0}(\infty)$ and $\frac{\partial Q_{i_0}(\infty)}{\partial \alpha_{i_1}}$, using $(14 - 15)$, $(21 - 23)$ and (24).

As for the transitory quantities, one needs to compute numerically quantities of the shape $\rho_t h$ and $\varphi_t^{(i_0)}(i_1, x, s)$ which appears in $Q_{i_0}(t)$ and $\frac{\partial Q_{i_0}(t)}{\partial \alpha_{i_1}}$ (see $(16 - 17)$ and $(18 - 20)$). Such quantities are here computed using finite volume methods as in Cocozza & co (2006-2).

6.3 Numerical example

We assume that the system initially is in state $(I_0, X_0) = (0, R/2)$. Beside, we take:

$$\lambda_0(x) = x^{\alpha_0} \quad ; \quad r_0(x) = (R - x)^{r_0} \quad ; \quad \lambda_1(x) = (R - x)^{\alpha_1} \quad ; \quad r_1(x) = x^{r_1}$$

for $x \in [0, R]$ with $\alpha_i > 0$ and $r_i > 1$. All conditions for irreducibility are here achieved.

We take the following numerical values:

$$\alpha_0 = 1.05; r_0 = 1.2; \alpha_1 = 1.10; r_1 = 1.1; R = 1; a = 0.3; b = 0.7.$$

Similarly as for the first method, we test our results using finite differences (FD). The results are here rather stable choosing different values for ε and the results are provided for $\varepsilon = 10^{-6}$. The asymptotic results are given in Table 2 and the transitory ones are given in Table 3 for $t = 2$.

Table 2: $\frac{\partial Q_{i_0}(\infty)}{\partial \alpha_{i_1}}$ by finite differences (FD) and the present method (MR).

i	$\frac{\partial Q_1(\infty)}{\partial \alpha_i}$		$\frac{\partial Q_2(\infty)}{\partial \alpha_i}$	
	FD	MR	FD	MR
0	-1.4471×10^{-2}	-1.4469×10^{-2}	-5.5294×10^{-2}	-5.5303×10^{-2}
1	-1.7471×10^{-2}	-1.7469×10^{-2}	-4.9948×10^{-2}	-4.9946×10^{-2}

Table 3: $\frac{\partial Q_{i_0}(t)}{\partial \alpha_{i_1}}$ for $t = 2$ by finite differences (FD) and the present method (MR).

i	$\frac{\partial Q_1(2)}{\partial \alpha_i}$		$\frac{\partial Q_2(2)}{\partial \alpha_i}$	
	FD	MR	FD	MR
0	-4.7561×10^{-2}	-4.7580×10^{-2}	-8.6747×10^{-2}	-8.6599×10^{-2}
1	-4.5566×10^{-3}	-4.5166×10^{-3}	-2.7299×10^{-2}	-2.7370×10^{-2}

The results are very similar by FD and MR both for asymptotic and transitory quantities, which clearly validate the method.

7 Conclusion

This paper is a first step for the computation of derivatives of functionals of PDMP with respect to some parameter p. The results appear as extensions of those available for jump Markov processes in Cao and Chen (1997) and Gandini (1990). However, the present study is restricted to the case where only the discrete transition rates depend on parameter p. Some additional mathematical work remains to be done to extend the results to more general cases. Also, some reflexion should be lead on to go through the numerical computation of the importance functions, in case of larger studies than the small examples of this paper.

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