OPTIMAL RESTARTING DISTRIBUTION AFTER REPAIR FOR A MARKOV DETERIORATING SYSTEM

LOI OPTIMALE DE REDEMARRAGE APRES REPARATION POUR UN SYSTEME A DEGRADATION MARKOVIENNE

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Summary

We consider a repairable system such that different completeness degrees are possible for the repair (or corrective maintenance), that go from a 'minimal' up to a 'complete' repair. Our problem then is to find the optimal degree for the repair, namely such that the long-run availability is optimal. The system evolves in time according to a Markov process with a finite state space as long as it is running, whereas duration of repairs follow general distributions. After repair, the system starts again in an up-state *i* with the probability *d(i)*. This distribution *d* is called the "restarting distribution". Amazingly, we observe on an example that the optimal restarting distribution may be random, which highly complicates both of its research and achievement. Sufficient conditions under which this optimal restarting distribution is non random are then given. The optimal restarting distribution is provided for two classical structures in reliability (*k*-out-of-*n* structures and standby structures), as well as the optimal number of redundant components to be set up in such structures in case of complete repairs.

Résumé

Nous considérons un système réparable, pouvant être réparé de manière plus ou moins complète lorsqu'il tombe en panne. Notre problème est alors de trouver le degré optimal de réparation, le critère utilisé étant la disponibilité asymptotique. Tant qu'il est en marche, le système évolue selon un processus markovien à espace d'états fini, les durées de réparations suivant en revanche des lois générales. A l'issue d'une réparation, le système redémarre dans un état de marche *i* avec la probabilité *d(i)*. Nous disposons ainsi d'une "loi de redémarrage" *d*. De façon étonnante, nous observons sur un exemple que la loi optimale de redémarrage peut être aléatoire, ce qui en complique notablement à la fois la recherche et la réalisation. Nous donnons alors des conditions suffisantes pour qu'elle soit au contraire déterministe. A titre d'illustration, nous étudions deux structures classiques en fiabilité (structures de type *k-sur-n* et à redondance passive) pour lesquelles nous déterminons la loi optimale de redémarrage ainsi que le nombre optimal de composants à installer dans le système dans le cas où les réparations sont complètes.

Introduction

Let us consider a repairable system such that different completeness degrees are possible for the repair (or corrective maintenance), that go from a 'minimal' up to a 'complete' repair. One may think for instance of a system with redundant components. Our questions are: in case of failure, is it worth achieving complete repairs, that may be long (or costly), or is it better to repair the system as quickly as possible ? To which extent should the corrective maintenance be performed ? The answer to such questions highly depends on the criterion used to measure the performance of the system: we are interested here in the long-run availability, that is the probability for the system to be up in the long run. Our problem then is to find the degree of the repair such that the long-run availability is optimal. In [3], we already studied such kinds of problems but we concentrated there on finding conditions under which complete repairs are optimal. Namely, we showed that for a system with some kind of aging property, complete repairs are optimal. As for other papers about maintenance optimization, most of them actually deal with preventive maintenance, only a few with corrective maintenance. Nearest problems from ours may be found in papers dealing with redundancy optimization. One may think for instance to [1], to Chapter 6 of [2], or to [9] and references therein. In such papers, the authors are mainly interested in optimizing reliability under constraints or under the assumption of two failure modes. Their aim is to provide algorithms for finding optimal redundancy. The closest work from ours we found is [7] where the authors consider a system composed of *N* identical parallel units, for which they show (among other results) that even in the case of units with constant failure rate, cost may be improved by deliberately taking out of operation some non-failed units. The question then arises to find the optimal number of units to be put into operation (or to repair in case of failure), which they compute under different assumptions.

Here, we do not fix the structure of the system as the previous authors did, but we assume that the system evolves in time according to a Markov process as long as it is running. When the system fails, a repair is begun with a general distribution. After repair, we assume that the system always starts again in the same way. More precisely, if the up-states of the system are denoted by 1, 2, ..., *m*, the system starts again after any repair in state *i* ($1 \leq i \leq m$) with the same probability, denoted by *d*(*i*). This means that we allow the new starts after repair to be random. As for the technical realization of such a thing, let us think for instance of a system composed of two parallel sub-systems with two repairmen facilities. In case of failure, the repair of both sub-systems is begun simultaneously. Then, we may decide to let the system start again as soon as one is over. Also, we may adjust the new start of the system according to the desired restarting distribution by adding some repairmen facilities for one of the sub-systems.

For such a system (see next section for more details) we compute the long-run availability A_∞ (see Theorem 1) and then comes our problem, namely to look for the restarting distribution d_{opt} that makes the long-run availability optimal. We first observe from a numerical example that this optimal distribution does not always correspond to a new start in a fixed up-state and may be random. This justifies the introduction of a random distribution for the new starts after repair. Though, we also observe that the optimal distribution often is non random. A natural problem then is to look for conditions under which the optimization may be limited to such non random distributions. Indeed, from a practical point of view, it is easier to know exactly which components to repair in case of failure. Besides, from a theoretical point of view, the research of the optimal restarting distribution is, under such conditions, highly simplified. Indeed, there are, in that case, only m possible restarting distributions, whereas all the possible distributions on {1,...,m} have to be considered in the general case. Such sufficient conditions are given in Theorem 2. They are tested on some examples, and then used to study 'k-out-of-n' and standby structures: for both of them, we show that the optimal restarting distribution is non random and corresponds to an optimal number of components to be repaired in case of failure, which we compute. This easily provides us with the optimal number of redundant components to be set up in those structures, in case of complete repairs.

The proofs of the different results of this presentation may be found in [4]. Also, one may find in [3] some sufficient conditions for complete repair to be optimal, which are not exposed here.

We now specify our assumptions and notations.

Assumptions - Notations

The system evolves in a finite state space, composed of m up-states (1, 2, ..., m) and p down-states (m+1, ..., m+p). We assume that the system starts from an up-state and then evolves in time according to a Markov process up to its first failure. (Typically, we are thinking about a system formed of components with constant failure rates). This system almost surely breaks down after a finite time: $\Pr_i(T<+\infty) = 1$ for every i in {1,...,m}, where T is the first up-period of the system and $\Pr_i$ is the conditional probability given that the system started in state i. (The associated conditional expectation is denoted by E_i). The system evolves according to the same Markov process after any repair. The repair of the system begins as soon as it breaks down and has a random duration that is independent of the previous evolution of the system. If the system is in the down-state $m+k$ ($1\le k \le p$), and if the system starts again in the up-state i after repair, then the repair has the same (general) distribution as a random variable $R_{m+k,i}$, with a finite mean $E(R_{m+k,i})$. Let R be the pXm matrix of the $E(R_{m+k,i})$'s. After any repair, the system starts again in an up-state that is assumed to be independent of the previous evolution of the system (and consequently, on the down-state by the time of the repair). Then, $d(i)$ is the probability for the system to start again from state i ($1 \le i \le m$) after any repair and $d = d(1, 2, \dots, m)$ is the so-called restarting distribution (after repair), see Figure 1.

We denote by $(X_t)_{t\ge0}$ the Markov process that describes the evolution of the system up to its first failure:

$$X_t = \begin{cases} \text{state of the system} & \text{if } t < T, \\ m+k & \text{if } t \ge T \text{ and } X_T = m+k. \end{cases}$$

(The down-states of the system are made absorbing).

Let A be the (infinitesimal) generator of the Markov process (X_t), namely A is the matrix formed by the constant transition rates of (X_t) between the different states. The matrix A is subdivided as follows:

$$A = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline \bar{0}^{p,m} & \bar{0}^{p,p} \end{array}\right)$$

where $A_1 = (a_{i,j})_{1\le i,j\le m}$ (matrix of the transition rates between up-states), $A_2 = (a_{i,j})_{1\le i\le m, m+1\le j\le m+p}$ (matrix of the failure rates) and $\bar{0}_{p,m}$ (resp. $\bar{0}_{p,p}$) is the pXm (resp. pXp) matrix of zeros.

Let G be the mXm matrix such that $G_{i,j} = \int_0^{+\infty} P_t(i,j)\,dt$ where $P_t(i,j) = \Pr_i(X_t = j)$, for any $1 \le i,j \le m$, $t \ge 0$.

Then, $G_{i,j}$ represents the time spent in state j when the system starts from state i.

We recall that $G = A_1^{-1}$ (see [8] Theorem 4.25 for instance).

Finally, symbol $(Z_t)_{t\ge0}$ represents the process that describes the evolution of the system, with no truncation at time T. Also, for n in N*, symbols $\bar{1}_n$ and $\bar{0}_n$ respectively represent the nX1 column vectors filled with ones and zeros.

Computation of the Long-run Availability

According to our assumptions on the way the system starts again after repair and on the markovian evolution of the running system, it is clear that the later evolution of the system after a new start following a repair only depends on the up-state in which the system starts again and is independent of the past. Also, the successive states visited by the process $(Z_t)_{t\ge0}$ at each new start form a Markov chain. Consequently, the process $(Z_t)_{t\ge0}$ is what is called in literature a *semi-regenerative process*, with d as stationary distribution. For $1 \le i \le m$, symbols MUT_i and MDT_i now respectively represent the Mean Up Time and the Mean Down Time on a cycle of the semi-regenerative process $(Z_t)_{t\ge0}$ that starts in state i. Also, MUT_i and MDT_i respectively represent the mean time to failure and the mean duration of the first repair when the system starts from the up-state i. We use the following matricial notations:

$$\overline{MUT} = \begin{pmatrix} MUT_1 \\ MUT_2 \\ . \\ MUT_m \end{pmatrix} \text{ and } \overline{MDT} = \begin{pmatrix} MDT_1 \\ MDT_2 \\ . \\ MDT_m \end{pmatrix}.$$

Using general theorems from the Markov renewal theory, it is now easy to derive the following result (see [4] for details), where the numerator and the denominator of $a_\infty(d)$ respectively represent the Mean Up Time and the Mean Down Time of the system during a cycle of the semi-regenerative process $(Z_t)_{t\ge0}$.

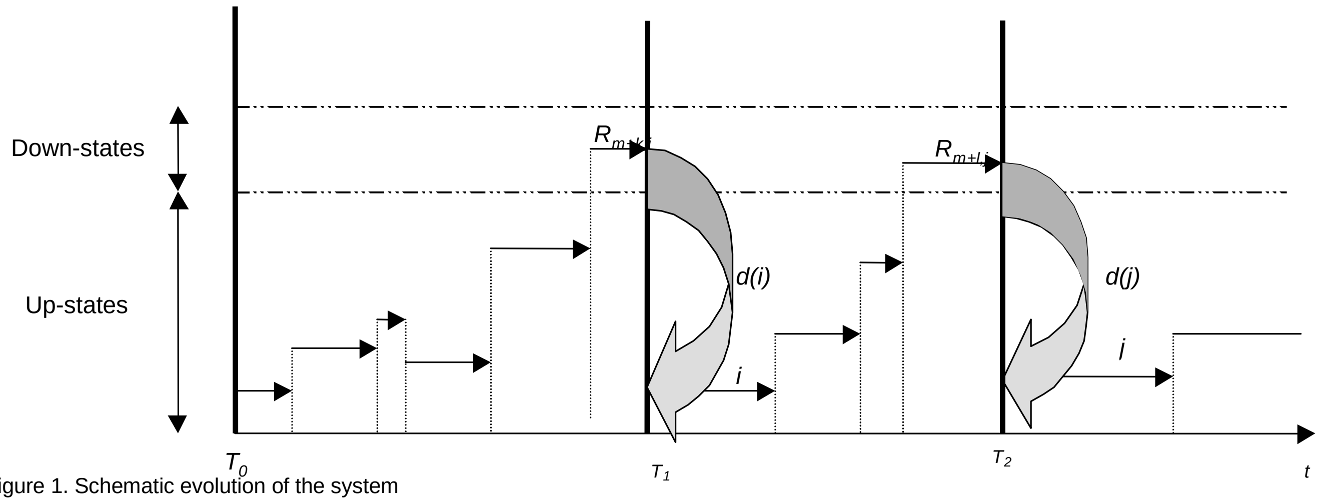

Figure 1. Schematic evolution of the system

Theorem 1. *The long-run availability of the system exists and is*

$$A_\infty(d) = \frac{1}{1+a_\infty(d)}$$

with

$$a_\infty(d) = \frac{d\times\overline{MDT}}{d\times\overline{MUT}} = \frac{dGA_2R\times^t d}{dG\bar{1}^m},$$

where $^t d$ is the transposed column vector of d.

A First Example

We now give a first example for which we look for the optimal distribution *d*.
For $1 \le i \le m$, δ_i represents the Dirac measure at *i* and $d = \delta_i$ corresponds to a new start in state *i*.

We consider a system formed of three components A, B and C, with respective constant failure rates λ_A, λ_B and λ_C. Component A has a constant repair rate μ_A. Component A and the sub-system constituted with components B and C in series are in standby redundancy (see Figure 2): at the beginning, component A is active and the subsystem is waiting. When component A fails, the sub-system is activated. Component B starts with probability $1 - \gamma_B$. Component C always starts.

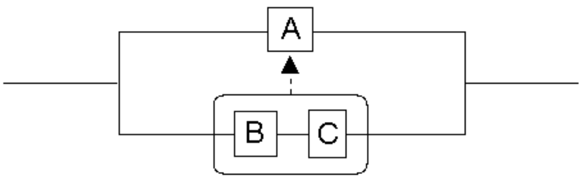

Figure 2. Structure of the system. Example 1.

We denote by 1 the perfect working state: $1=A(BC)_w$ (A is active, B and C are waiting) and 2 the second up-state: $2=\overline{A}$ BC (A is down, B and C are active).
There are two down-states: $3=\overline{A}\ \overline{B}$ C (A and B are down, C is up) and $4=\overline{A}$ B$\overline{C}$ (A and C are down, B is up), which may both lead to both up-states by repair.
We get:

$$A_1 = \begin{pmatrix} -\lambda_A & (1-\gamma_B)\lambda_A \\ \mu_A & -(\mu_A+\lambda_B+\lambda_C) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \gamma_B\lambda_A & 0 \\ \lambda_B & \lambda_C \end{pmatrix}.$$

We take $\lambda_A = 3$, $\lambda_B = 1$, $\lambda_C = 15$, $\gamma_B = 0.3$ and $\mu_A = 20$.
As for the repair, let us note that $R_{3,1}$, $R_{3,2}$, $R_{4,1}$ and $R_{4,2}$ respectively correspond to the repair of components A and B, of B, of A and C, of C. Besides, in the following examples, the repair of A is assumed to be quicker when the system is down than when it is up.
We take $d = [a,1\text{-}a]$ with a in [0,1] and we plot the long-run availability A_∞ with respect to *a* for different values of the mean repair lengths.

Case 1. We first assume that the mean repair of each component is 0.01 when the system is down, and that the different repairs may be undertaken simultaneously. Then we have:

$$R = \begin{pmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{pmatrix}.$$

and the long-run availability with respect to *a* is plotted in Figure 3.

Here, $A_\infty(d)$ is optimal for $d = [1,0] = \delta_1$ with $A_\infty(\delta_1) \approx 0.9830$.

Case 2. We now make the same assumptions apart from the fact that A and B may not be repaired simultaneously any more. Then we have

$$R = \begin{pmatrix} 0.02 & 0.01 \\ 0.01 & 0.01 \end{pmatrix}$$

And the long-run availability with respect to *a* is plotted in Figure 4.

Here, the optimal long-run availability is reached neither with δ_1, nor with δ_2. We get: $A_\infty(\delta_1) \approx 0.9721$, $A_\infty(\delta_2) \approx 0.9743$, $d_{\text{opt}} \approx [0.73,0.27]$, $A_\infty(d_{\text{opt}}) \approx 0.9746$.

Case 3. Finally, assume A and B may be repaired simultaneously, but not A and C. The mean duration for the repairs of A, B and C respectively are 0.015, 0.015, 0.01.

Then we have :

$$R = \begin{pmatrix} 0.015 & 0.015 \\ 0.025 & 0.01 \end{pmatrix}$$

The long-run availability with respect to *a* is plotted in Figure 5.

Here, $A_\infty(d)$ is optimal for $d_{\text{opt}} = \delta_2$, $A_\infty(\delta_2) \approx 0.9678$.

We observe from this example that the optimal restarting distribution d_{opt} may not always be chosen among the Dirac distributions (see the second case). There are however cases where it is possible (see the first and third cases, where the optimal distributions respectively are δ_1 and δ_2). We now give some conditions under which it is possible to limit the research for the optimal distribution among the Dirac ones.

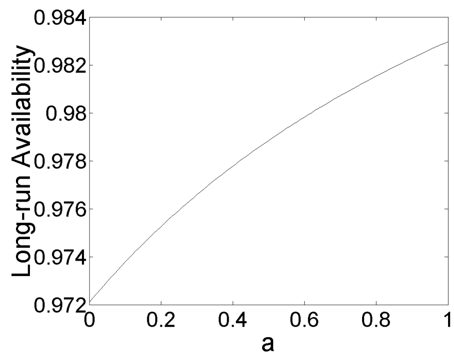

Figure 3. The long-run availability with respect to *a*, case 1.

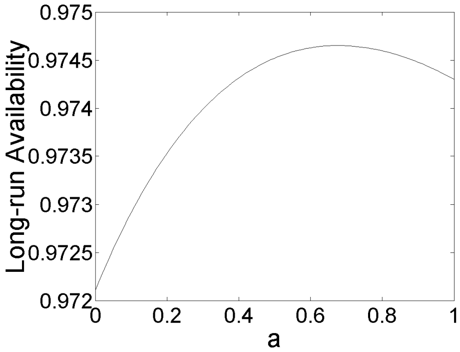

Figure 4. The long-run availability with respect to *a*, case 2.

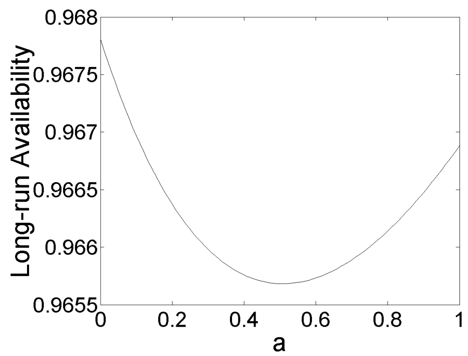

Figure 5. The long-run availability with respect to *a*, case 3.

Restriction of the Search for the Optimal Restarting Distribution to Dirac Distributions

Let us first note that, with some argument of continuity over a compact set, there clearly exists an optimal distribution d_{opt} (among all the possible restarting distributions) that makes the long-run availability optimal. Also, among the *m* different new starts in a fixed up-state *i* (which corresponds to $d_{\text{opt}} = \delta_i$), one of them, say δ_{io}, is better than the others, in the sense that $A_\infty(\delta_i) \leq A_\infty(\delta_{io})$ for any $1 \leq i \leq m$. We now give conditions for δ_{io} to be optimal among all the possible restarting distributions *d* and not only among Dirac distributions.

Theorem 2. *Let (H_1) and (H_2) be the following assumptions:*
(H_1) For any fixed k such that $2 \leq k \leq p$, $E(R_{m+k,i}) - E(R_{m+k-1,i})$ is independent of i for $1 \leq i \leq m$.
(H_2) For any fixed k such that $2 \leq k \leq p$,

$(E(R_{m+k,i}) - E(R_{m+k-1,i}))_{1\leq i\leq m}$ and $(\sum_{l=k}^{p}(GA_2)(i,l))_{1\leq i\leq m}$ are monotone with respect to i, in opposite directions.
Then, under (H_1) OR (H_2), there is a non random restarting distribution optimal among all the possible restarting distributions. Namely: if io ($1 \leq io \leq m$) is such that $A_\infty(\delta_{io}) = \max_{1\leq io\leq m} A_\infty(\delta_i)$, we then have $A_\infty(d) \leq A_\infty(\delta_{io})$ for any distribution d on $\{1,...,m\}$.

(The proof may be found in [4]).

We now indicate a few situations in which assumption (H_1) clearly is true, so that we may see clearer when the previous result may be applied.

Assumption (H_1) is clearly true in the following situations:

- There is one single down state or, more generally, the duration of the repair is independent of *m+k* and *i*. This will happen for instance if the repairman has to be called in case of failure and if the duration of the repair itself is negligible relative to the waiting time for the repairman to arrive.
- The duration of the repair is independent of the down state at the time of the repair (i.e. $E(R_{m+k,i}) - E(R_{m+k-1,i})$ for any $1\leq i \leq m$, $2 \leq k \leq p$). This is true, for instance, when the repairs of the components necessary to the good working of the system are short relative to the others.
- The duration of the repair is independent of the state of the system after repair (i.e. $E(R_{m+k,i}) = E(R_{m+k,i+1})$ for any $1 \leq i \leq m-1$, $1 \leq k \leq p$). This is true, for instance, when the repairs of the components necessary to the good working of the system are long relative to the others.
- There is only one single repairman facility so that the duration of the repair is the addition of the duration of the repairs of the different components.

As for the meaning of assumption (H_2), we may note that if the up-states are ranked according to their increasing degradation degree, $(\sum_{l=k}^{p}(GA_2)(i,l))_{1\leq i\leq m}$ often is increasing. Indeed, this property means that, if the system starts from state *i*, the first visited down-state is stochastically smaller than if the system starts from state *i*+1 and may be understood as some kind of ageing property (see [3] for details). This is often true. Then, the most restrictive part of (H_2) is the assumption on the duration of repairs.

Let us now check the conditions given by Theorem 2 on Example 1 given in the previous section, and on a new example.

Back to Example 1

Condition $2 \leq k \leq p$ here reduces to $k = 2$ and we have

$$((gA_2)(i,2))_{1\leq i\leq 2} = \begin{pmatrix} 0.4773 \\ 0.6818 \end{pmatrix}.$$

Then, we are in the case where $(\sum_{l=k}^{p}(GA_2)(i,l))_{1\leq i\leq m}$ increases with *i*.

In the first case, $(E(R_{m+k,i}) - E(R_{m+k-1,i}))_{1\leq i\leq m}$ is independent of *i*, so that (H_1) is true. The optimal distribution is non random: $d_{\text{opt}} = \delta_1$.

In the second case, $(E(R_{m+k,i}) - E(R_{m+k-1,i}))_{1\leq i\leq m}$ increases with *i*, so that neither (H_1) nor (H_2) is true. The optimal distribution is random: $d_{\text{opt}} \approx [0.73, 0.27]$.

In the third and last case, $(E(R_{m+k,i}) - E(R_{m+k-1,i}))_{1\leq i\leq m}$ decreases with *i* and (H_2) is true. The optimal distribution is non random $d_{\text{opt}} = \delta_2$.

Example 2

We consider a system formed of three components A, B and C, with respective constant failure rates λ_A, λ_B and λ_C. These components cannot be repaired when the system is active. Component A and the subsystem composed with components B and C in parallel are in series (see Figure 6). The failure of component A brings about the failure of components B and C as well.

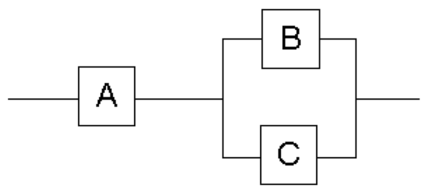

Figure 6. Structure of the system. Example 2.

Using the same notations as for Example 1, let 1 = ABC, 2 = $AB\overline{C}$ and 3 = $A\overline{B}C$ be the up-states, and 4 = $A\overline{B}\,\overline{C}$ and 5 = $\overline{A}\,\overline{B}\,\overline{C}$ be the down-states. Both down states may lead by repair to the three up-states ($m = 3$).

$$A_1 = \begin{pmatrix} -(\lambda_A + \lambda_B + \lambda_C) & \lambda_C & \lambda_B \\ 0 & -(\lambda_A + \lambda_B) & 0 \\ 0 & 0 & -(\lambda_A + \lambda_C) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & \lambda_A \\ \lambda_B & \lambda_A \\ \lambda_C & \lambda_A \end{pmatrix}$$

Case 1. Components B and C may be repaired simultaneously, or components A and C, but not components A and B. The mean repair-time for each component A, B and C is 0.005. We get :

$$R = \begin{pmatrix} 0.005 & 0.005 & 0.005 \\ 0.01 & 0.01 & 0.005 \end{pmatrix}$$

and $(E(R_{5,i}) - E(R_{4,i}))_{1\le i\le m}$ decreases with *i.*

We take λ_A =1.8, λ_B =0.05 and λ_C = 1.85 and we get :

$$((gA_2)(i,2))_{1\le i\le 3} = \begin{pmatrix} 0.9796 \\ 0.9730 \\ 0.4932 \end{pmatrix}.$$

Then $(\sum_{l=k}^{p}(GA_2)(i,l))_{1\le i\le m})$ decreases with *i* and *(H₂)* is false.

We take *d* = [a,b,1-a-b] (with a ≥ 0, b ≥ 0, a+b ≤ 1) and the long-run availability is plotted with respect to a and b in Figure 7.

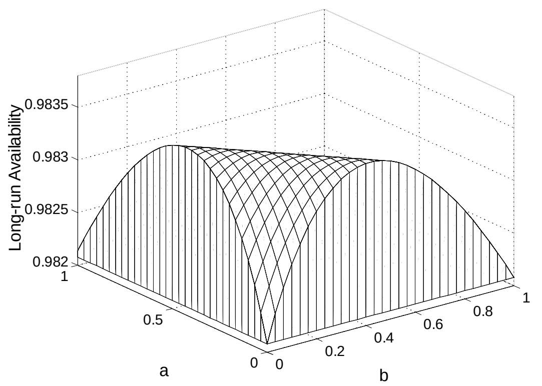

Figure 7. The long-run availability with respect to a and b, case 1.

The long-run availability is optimal for d_{opt} ≈ [0.4202,0,0.5798] and $A_\infty(d_{opt})$ ≈ 0.9836. Moreover, $A_\infty(\delta_1) \approx A_\infty(\delta_2) \approx A_\infty(\delta_3) \approx$ 0.9821.
Here, δ_1, δ_2 and δ_3 correspond to minima of A_∞: *any random restarting distribution is better than a deterministic one!*

Case 2. Components A and B may be repaired simultaneously, but not components B and C, or components A and C. The mean repair-times for components A, B and C respectively are 0.02, 0.02 and 0.001. We get :

$$R = \begin{pmatrix} 0.021 & 0.02 & 0.001 \\ 0.021 & 0.02 & 0.021 \end{pmatrix}$$

and $(E(R_{5,i}) - E(R_{4,i}))_{1\le i\le m}$ increases with i.

We take λ_A =5, λ_B =1 and λ_C =4 and we get :

$$((gA_2)(i,2))_{1\le i\le 3} = \begin{pmatrix} 0.8889 \\ 0.8333 \\ 0.5556 \end{pmatrix}$$

and $(\sum_{l=k}^{p}(GA_2)(i,l))_{1\le i\le m})$ decreases with *i.* Then, *(H₂)* is true.

Here again, we take *d* = [a,b,1-a-b] and the long-run availability is plotted with respect to a and b in Figure 8.
The long-run availability is optimal for $d_{opt} = \delta_3$ and $A_\infty(\delta_1) \approx$ 0.8944, $A_\infty(\delta_2) \approx$ 0.8929, $A_\infty(\delta_3) \approx$ 0.9017.

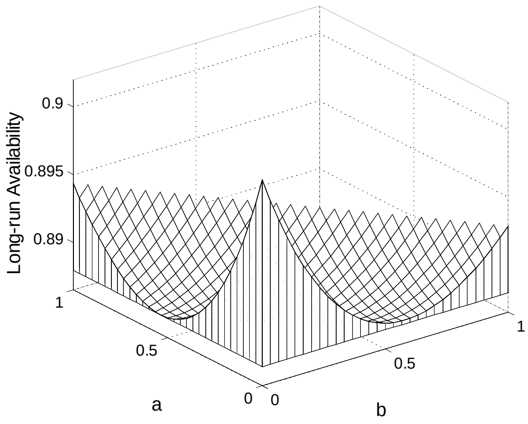

Figure 8. The long-run availability with respect to a and b, case 2.

Two Classical Structures in Reliability

We here consider two classical structures in reliability. In both cases, the system is composed of *n* identical and independent components with constant failure rate λ >0.

The first structure is what is called a '*k*-out-of-*n*' one: the system is up if and only if at least *k* components are up (1 ≤ *k* ≤ *n*).

In the second structure, the components are in standby redundancy: one single component is active at a time, the redundant units are standing by as spares and used successively for replacement. When activated, a component starts successfully with probability 1- γ (with 0 < γ <1).

For both structures, no repair may be performed as long as the system is active. In case of failure, a repairman is called. The mean waiting time until his arrival is *c*, the mean duration for the repair of one component is *r* (*c*, *r* ≥ 0).

We first look for the optimal restarting distribution. As there is, in both cases, one single down-state, we may apply Theorem 2, so that we only have to search among Dirac distributions. Once we have determined this optimal restarting distribution (or, equivalently, the optimal number of components to be up after repair), we easily derive the optimal number of components to be set up in the system, with *c*, *r*, λ and *k* (respectively *c*, *r*, λ and *γ*) fixed in the first (respectively second) case, in case of complete repairs.

Case of a '*k*-out-of-*n*' structure

Let *i* be the state where exactly *i* - 1 components are failed (1 ≤ *i* ≤ *n*). There are here *m* = *n* - *k* + 1 up-states. The single down-state (*m* + 1 = *n* - *k* + 2) corresponds to *n* - *k* + 1 failed components.
According to the assumptions concerning the repair (see the introduction of this section), we get: $E(R_{m+1,i}) = c + (n-i-k+2)r$, where *n* - *i* - *k* + 2 is the number of components to repair (1 ≤ *i* ≤ *m* = *n* - *k* + 1).

We get the following results:

Proposition 3.
1. *For k and n fixed (1 ≤ k ≤ n) :*
   a. *If r=0 or n = k, the best is to repair every component.*
   b. *If* $r \neq 0$ *and n > k:*

Table 1. Optimal number of components n_0 according to *c*, case *k* = 5, *r* = 0.1.

c	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.60	0.65	
n_0	5	6	7	8	9	10	10	11	11	12	12	13	13	14	
c	0.7	0.75	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2	1.25	1.3	1.35	1.4
n_0	14	14	15	15	16	16	16	17	17	18	18	18	19	19	19

- *If c/r ≤ 1/k, the best is to repair one single component.*
- *If* $\frac{c}{r} \geq \sum_{j=k}^{n-1}\left(\frac{n}{j}-1\right)$ *: the best is to repair every component.*
- *If* $\frac{1}{k} < \frac{c}{r} < \sum_{j=k}^{n-1}\left(\frac{n}{j}-1\right)$ *: the optimal restarting distribution is* δ_{i_0} *(2 ≤* i_0 *≤ m-1), which means that we have to repair n -* i_0 *- k + 2 components, where* i_0 *is the unique integer such that*

$$\sum_{j=k}^{n-i_0}\left(\frac{n-i_0+1}{j}-1\right) \leq \frac{c}{r} \leq \sum_{j=k}^{n-i_0+1}\left(\frac{n-i_0+2}{j}-1\right)$$

(i_0 *independent of* λ)

2. *For k fixed, let* n_0 *be the optimal number of components tobe set up. We get:*
   a. *If r=0: then* n_0 *= ∞ (we have to set up as many components as possible).*
   b. *If r* $r \neq 0$ *:*
      - *If c=0: then* n_0 *=k and redundant components *are prejudicial to the long-run availability.*
      - *If c* $r \neq 0$ *0: then* n_0 *is the unique integer such that*

$$\sum_{j=k}^{n_0-1}\left(\frac{n_0}{j}-1\right) \leq \frac{c}{r} < \sum_{j=k}^{n_0}\left(\frac{n_0+1}{j}-1\right)$$

(n_0≥ k, independent on λ).

A specific case: For a 5 out of *n* system and for *r* = 0.1, the optimal number of components to set up in case of complete repair is given in Table 1 according to the value of *c*.

Case of *n* components in standby redundancy

Let *i* be the state where exactly *i* -1 components have failed (1 ≤ *i* ≤ *n*). There are *m* = *n* up-states. The system is down when all the *n* components are failed (*m*+1= *n*+1).

According to the assumptions concerning the repair, we have: $E(R_{m+1,i}) = c + (n+1-i)r$ for any 1 ≤ *i* ≤ *n*.

We get:

Proposition 4.

1. *For n fixed:*
   - *If r <((1 - γ)/ γ)c (independent condition on λ and on n): the best is to repair every component.*
   - *If r ≥((1 - γ)/ γ)c: the best is to repair one single component.*
2. *We can derive:*
   - *If r <((1 - γ)/ γ)c: we have to set up as many components as possible.*
   - *If r ≥((1 - γ)/ γ)c: the best is to set up one single component and redundant components are prejudicial to the long-run availability.*
   -

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