Optimization of the Corrective Maintenance of a Repairable System

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Abstract. We consider a repairable system such that different completeness degrees are possible for the repair (or corrective maintenance), that go from a 'minimal' up to a 'complete' repair. Our questions are: to what extent must the system be repaired in case of failure for the long-run availability to be optimal? In which cases are complete repairs optimal? The system evolves in time according to a Markov process as long as it is running whereas the durations of repairs follow general distributions. After repair, the system starts again in an up-state i with probability $d(i)$. We give conditions under which the optimal restarting distribution d^{opt} is non random. Besides, we show that, for an 'ageing' system, the more complete the repair, the higher the stationary availability. The 'ageing' property of the system is expressed with some monotonicity for the underlying Markov process with respect to the reversed hazard rate ordering.

1 Introduction

Let us consider a repairable system such that different completeness degrees are possible for the repair (or *corrective maintenance*), that go from a 'minimal' up to a 'complete' repair. Then, in case of failure, is it worth achieving complete repairs, that may be long (or costly), or is it better to repair the system as quickly as possible ? To which extent should the corrective maintenance be performed ? In which cases are complete repairs optimal ?

The criterion used to measure the performance of the system is the long-run availability, namely the probability for the system to be up when in long time run. Besides, the system is assumed to evolve in time according to a Markov process with a finite state space up to its first failure, and in the same way after any repair. It is subject to different kinds of failure. To each corresponds a repair with a random duration and a general distribution. After any repair, the new start of the system is independent on the previous evolution of the system and is controlled by a fixed distribution d on the up-states: if the up-states of the system are denoted by 1, 2, ..., m, the system starts again after any repair in state i $(1 \leq i \leq m)$ with probability $d(i)$.

For such a system, we compute the long-run availability $A_\infty(d)$ (see Section 3) and then look for the optimal restarting distribution d^{opt}, namely such that the long-run availability is optimal. It can be seen on numerical examples (see Bloch-Mercier 2000, 2001b) that this optimal distribution does not always correspond to a new start in a fixed up-state and may be random. From a theoretical point of view, this highly complicates its research. From a practical point of view, it is also easier to know exactly which component to fix in case of failure. Both of those remarks have lead us to study conditions under which the optimal restarting distribution is non random, which are given in Section 4.

Besides, we may note that complete repairs are usual in industry. Indeed, the opposite situation means that there are some useless components which are never repaired. (Though, note that some economical or technical constraints may lead to set them up nevertheless). A natural question then is: under which conditions are complete repairs optimal ? Such conditions are given in Section 5: for a system with some kind of 'ageing' property, if it takes the same time to achieve a complete or a minimal repair, we show that the long run availability is all the higher as the repair is complete, so that complete repairs are optimal. The 'ageing' property of the system is translated through some monotonicity for the underlying Markov process with respect to the reversed hazard rate ordering. The degree of completeness of the repair is measured with the reversed hazard rate ordering too.

We first give in Section 2 some preliminary results on $rh-$monotone Markov processes from where the results of Section 5 are derived.

2 Preliminary

Let ν_1 and ν_2 be two probability vectors on $\{1,...,m\}$. We recall that ν_1 is said to be greater than ν_2 in the sense of reversed hazard rate ordering, denoted by $\nu_1 \prec_{rh} \nu_2$, if and only if

$$\frac{\sum_{k=1}^{j} \nu_1(k)}{\sum_{k=1}^{i} \nu_1(k)} \leq \frac{\sum_{k=1}^{j} \nu_2(k)}{\sum_{k=1}^{i} \nu_2(k)}, \text{ for any } 1 \leq i \leq j \leq m,$$

when defined, using the convention $\frac{0}{0} = 0$ (see Keilson and Sumita (1982), Shaked and Shanthikumar (1994), Kijima (1997) or Block, Savits and Singh (1998) for instance).

Now, let (Y_t) be a Markov process on the finite state space $\{1,...,m+1\}$, let $(P'_t(i,j))_{1\leq i,j\leq m+1}$ be the associated semi-group ($P'_t(i,j) = \mathbb{P}_i(Y_t = j)$ for any $1 \leq i,j \leq m+1$ and $t \geq 0$), and let $A' = \left(a'_{i,j}\right)_{1\leq i,j\leq m+1}$ be its (infinitesimal) generator.

Also, for $1 \leq i \leq m+1$, let $P'_t(i,\bullet)$ be the i-th row of $(P'_t(i,j))_{1\leq i,j\leq m+1}$.

Following Kijima (1998), we first recall the definition of an rh-monotone Markov process and its characterization in terms of its generator:

Definition 1. The process (Y_t) is an rh-monotone Markov process if and only if $P'_t(i,\bullet) \prec_{rh} P'_t(i+1,\bullet)$ for any $1 \leq i \leq m$, $t \geq 0$.

Theorem 1 (Kijima (1998)). *The process* (Y_t) *is an* rh*-monotone Markov process if and only if*

$$a'_{i,j} = 0 \text{ for any } 1 \le j \le i-2 \le m+1$$
$$\text{and } a'_{i,j} \le a'_{i+1,j} \text{ for any } 3 \le i+2 \le j \le m+1.$$

We now give two other results for rh-monotone Markov processes with upper triangular generators. Some interpretation of the first result may be found in the few lines following Theorem 6.

Theorem 2. *Let* (Y_t) *be an* rh*-monotone Markov process. Then:*

$$\left(P'_t(i,\bullet) \prec_{rh} P'_s(i,\bullet) \text{ for any } 1 \le i \le m+1,\ 0 \le t \le s\right)$$
$$\Longleftrightarrow (A' \text{ is upper triangular})$$

Theorem 3. *Let* (Y_t) *be an* rh*-monotone Markov process with an upper triangular generator. Then:*

$$\frac{\sum_{j=1}^{k+1} \int_0^{+\infty} P'_t(i,j)\,dt}{\sum_{j=1}^{k} \int_0^{+\infty} P'_t(i,j)\,dt} \le \frac{\sum_{j=1}^{k+1} \int_0^{+\infty} P'_t(i+1,j)\,dt}{\sum_{j=1}^{k} \int_0^{+\infty} P'_t(i+1,j)\,dt} \tag{1}$$

for any $1 \le i \le k \le m$.

Note that, without integral signs, inequalities (1) simply translate the rh-monotonicity of the process (Y_t). The problem is to add the integral signs, which is done under the additional assumption of an upper triangular generator.

We now come to our reliability problem.

3 Computation of the Long-run Availability

Let 1, 2, ..., m and $m+1$, ..., $m+p$ respectively be the up and down states of the system. The symbol $\mathbb{E}(R_{m+k,i})$ represents the mean duration of the repair associated to the down-state $m+k$ ($1 \le k \le p$) that puts the system back to the up-state i ($1 \le i \le m$). Also, let $R = (\mathbb{E}(R_{m+k,i}))_{1\le k\le p, 1\le i\le m}$. Let us recall that $d(i)$ is the probability for the system to start again from state i ($1 \le i \le m$) after *any* repair and let d be the probability vector $d = (d(1), d(2), ..., d(m))$.

Let T be the first on-period of the system and let (X_t) be the *Markov process* that describes the evolution of the system up to its first failure:

$$X_t = \begin{cases} \text{state of the system if } t < T, \\ m+k \qquad\qquad\quad \text{if } t \ge T \text{ and } X_T = m+k. \end{cases}$$

Let $(P_t(i,j))_{1\le i,j\le m}$ be the semi-group associated to (X_t) and let $A = (a_{i,j})_{\substack{1\le i\le m+p\\ 1\le j\le m+p}}$ be its (infinitesimal) generator. We also use the following submatrices of A: $A_1 = (a_{i,j})_{\substack{1\le i\le m\\ 1\le j\le m}}$ and $A_2 = (a_{i,j})_{\substack{1\le i\le m\\ m+1\le j\le m+p}}$.

Let $G_{i,j} = \int_0^{+\infty} P_t(i,j)\,dt$ for any $1 \leq i,j \leq m$ and $G = (G_{i,j})_{\substack{1\leq i\leq m \\ 1\leq j\leq m}}$. Symbol $G_{i,j}$ represents the time spent in state j before failure when the systems starts from state i. We recall that $G = -A_1^{-1}$ (see Kijima 1997, Theorem 4.25 for instance).

Finally, let $\bar{1}^n$ be the $n \times 1$ column vector of ones (with $n \in \mathbb{N}^*$).

We get the following result:

Theorem 4. *The long-run availability of the system exists and is*

$$A_\infty(d) = \frac{1}{1 + a_\infty(d)}$$

with

$$a_\infty(d) = \frac{dGA_2R\left({}^t d\right)}{dG\bar{1}^m}, \tag{2}$$

where ${}^t d$ *is the column vector transposed of* d.

The computation of the long-run availability is based on the fact that the later evolution of the system after a new start following a repair only depends on the state in which the system starts again after repair. If (Z_t) is the process that describes the evolution of the system, with no truncation at time T, the previous remark then means that (Z_t) is a *semi-regenerative process* so that we may apply general theorems from the Markov renewal theory (see Çinlar (1975) or Cocozza-Thivent (1997) for instance). The numerator and denominator of $a_\infty(d)$ now respectively represent the Mean Down Time and Mean Up Time of the system on a cycle of (Z_t).

4 Restriction of the Research of the Optimal Restarting Distribution to Dirac Distributions

For $1 \leq i \leq m$, let δ_i be the Dirac measure at i. Then, among the m different new starts in a fixed up-state i (which corresponds to $d = \delta_i$), there clearly exists one, say δ_{i_0}, such that $A_\infty(\delta_i) \leq A_\infty(\delta_{i_0})$ for any $1 \leq i \leq m$. We now give conditions for δ_{i_0} to be optimal not only among the "deterministic" new starts but also among all the possible restarting distributions d.

Theorem 5. *Let* (H_1) *and* (H_2) *be the following assumptions:*

(H_1) *For any fixed* k *such that* $2 \leq k \leq p$, $\mathbb{E}(R_{m+k,i}) - \mathbb{E}(R_{m+k-1,i})$ *is independent of* $i\ (1 \leq i \leq m)$.

(H_2) *For any fixed* $k\,(2 \leq k \leq p)$, $\left(\mathbb{E}(R_{m+k,i}) - \mathbb{E}(R_{m+k-1,i})\right)_{1\leq i\leq m}$ *and* $\left(\sum_{l=k}^{p}(gA_2)(i,l)\right)_{1\leq i\leq m}$ *are monotone with respect to* i, *with opposite variations.*

Then, under (H_1) ***or*** (H_2)*, there is a* ***non random*** *restarting distribution optimal among all the possible restarting distributions: if* i_0 $(1 \leq i_0 \leq m)$ *is such that* $A_\infty(\delta_{i_0}) = \max_{1\leq i\leq m} A_\infty(\delta_i)$*, we then have* $A_\infty(d) \leq A_\infty(\delta_{i_0})$ *for any distribution* d *on* $\{1,...,m\}$*.*

Note that no condition is required for the previous result to be true in case of a single down-state. Also, it is easy to see that (H_1) is true when durations of repairs are independent on $m+k$ **or** on i, or when there is only one single repairman facility so that the duration of repair simply is the addition of the durations of repairs for the different components.

As for assumption (H_2), one may easily check that the most restrictive part concerns the mean durations of repairs.

5 A Sufficient Condition for Complete Repairs to be Optimal

We now give sufficient conditions for complete repairs to be optimal (with respect to the long-run availability), or more generally, for the long-run availability to be all the higher as the repair is complete. Those conditions are based on the preliminary results given in Section 2. The process (Y_t) here represents the evolution of the system up to its first failure, where the down-states have been aggregated:

$$Y_t = \begin{cases} \text{state of the system} & \text{if } t < T, \\ m+1 & \text{if } t \geq T. \end{cases}$$

The mean duration for the repair $\mathbb{E}(R_{m+k,i})$ is here assumed to be independent on i and we now note $\mathbb{E}(R_{m+k,i}) = \mathbb{E}(R_{m+k})$. Also, r is the column vector of the $\mathbb{E}(R_{m+k})$'s for $1 \leq k \leq p$.

Theorem 6. *Let us assume that :*

(H'_1) $A_2 r$ *is increasing componentwise,*
(H'_2) (Y_t) *is* rh*-monotone with an upper triangular generator (which is equivalent to* $A_2\bar{1}^p$ *is increasing componentwise and* A_1 *is upper triangular such that* $a_{i,j} \leq a_{i+1,j}$ *for any* $3 \leq i+2 \leq j \leq m$*).*

Then, for any probability row vectors d_1 *and* d_2 *on* $\{1,...,m\}$*:*

$$d_1 \prec_{rh} d_2 \Longrightarrow D_\infty(d_1) \geq D_\infty(d_2).$$

In particular, the long-run availability is optimal for complete repairs.

Assumptions '$A_2 r$ and $A_2\bar{1}^p$ are increasing componentwise' respectively mean that the mean duration of the repair following a breakdown in state i and that the 'global' failure rate associated to state i $(\sum_{j=1}^{p} a_{i,m+j})$ are both increasing with i for $1 \leq i \leq m$. Consequently, they mean that *the up-states are ranked according to their increasing degradation degree.*

Besides, according to Theorem 2, (H_2') means that $P_t(i, \bullet) \prec_{rh} P_s(i, \bullet)$ for any $0 \leq t \leq s$, $1 \leq i \leq m$. Consequently, (H_2') is now equivalent to saying that the system is more degraded at time s than at time t, or that the system has got some 'ageing' property.

Finally, $d_1 \prec_{rh} d_2$ may be interpreted as 'the repair associated to d_1' is more complete than 'the repair associated to d_2'.

Now, Theorem 6 is the translation of the desired property, namely: the more complete the repair, the higher the long run availability.

Note that some examples may be found in Bloch-Mercier (2001) showing that this property is false if the ageing property of the system or the completeness degree of the repair are measured with the usual stochastic ordering, which justify the employment of the reversed hazard rate ordering.

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