Quasi Monte-Carlo methods for the numerical assessment of investments plans

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In order to assess investments plans, economic indicators need to be quantified. These indicators describe the expected gain as well as the economic risks. Monte-Carlo simulations are often used in this context. However, they require a large computational time to obtain accurate results. As our goal is to find an optimal strategy, Monte-Carlo simulations are not appropriate. Indeed, the Monte-Carlo method would require a too long computational time within an optimization algorithm. Here we propose to use quasi Monte-Carlo methods as an alternative, which provide accurate results more quickly than the Monte-Carlo method.

Keywords: Quasi Monte-Carlo methods, Markov chain simulation, Piecewise Deterministic Markov Process

1 Introduction

One goal of asset management is to optimize investments planning to improve the management of a fleet of components, for example in order to reduce down-times and thus creates added value. In the present context, an investments plan defines the dates of preventive maintenance actions and of spare parts orders. All components share a common stock of spare parts. A new investments plan is compared through economic indicators with a reference strategy, which does not consider any preventive maintenance action (no investments). These economic indicators are all linked to the Net Present Value (NPV), which stands for the difference between the cumulated discounted costs of both strategies, with and without investments. Hence the incomes are effective created profits (increased production) or avoided losses (reduced unavailability). A positive NPV reflects a reduction in operating and maintenance costs when considering the new investment plan in comparison with the reference strategy, while a negative NPV means that the investments costs are higher than savings. The NPV depends on the dates of component failures, and hence is a random variable. A first indicator for the assessment of an investments plan is the expected NPV. However, the expected NPV is insufficient to measure the economic venture linked to the new investment plan. Thus a second indicator is mandatory and we use the probability for the NPV to be negative, which stands for the probability to regret the performed investment. On Figure 1, the NPV probability density functions (p.d.f.) are plotted for two different strategies (NPV 1 and NPV 2). On this figure, we observe that the expected NPV is higher for strategy 2 than for strategy 1. However, strategy 2 has the highest probability for the NPV to be negative. According to the risk aversion of the decision maker, strategy 2 is hence not necessarily the one to be retained. Hence the point is to optimize an investments plan with respect to the expected NPV, under the constraint that the probability to regret the investment remains below a fixed threshold.

The quantification of the NPV p.d.f. requires the modelling of the stochastic evolution of the fleet of components and more specifically, the modelling of the evolution of the random costs associated to both strategies. Due to a common history up to the first time when the two strategies differ, the associated costs are correlated.

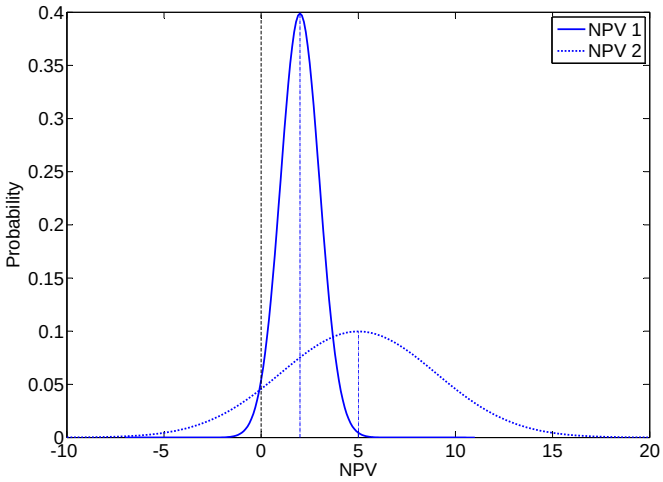

Fig. 1: Comparison of two probabilistic distributions of the NPV

Hence the NPV is the difference between two dependent random costs and the evaluation of its distribution first goes through an elaborate modeling of the fleet evolution and random costs, under both strategies, with and without investments. For this purpose, Piecewise Deterministic Markov Processes (PDMPs) [3] are used, which allow to catch the dependence between the two costs. It is possible to quantify the distribution of PDMP and associated quantities by finite volume methods, see [10], [11], [8]. However these methods require a large memory space and their programming becomes difficult when the number of components is greater than 3 or 4. As the referred industrial cases have more components (at least 10), these methods do not seem appropriate.

More classically Monte Carlo simulations are used to evaluate interest quantities. However Monte-Carlo simulations require a long computational time which may be prohibitive within an optimization algorithm. In this article, we propose two alternatives based on the quasi-Monte Carlo method, which consists on replacing uniformly distributed random variables by deterministic sequences, which have better uniformity properties.

2 Modelling the fleet of components and analysis of the Net Present Value probabilistic distribution

2.1 Modelling the fleet of components

A fleet of m identical components is considered. The cost function associated to an investments plan takes into account costs due to planned and unplanned replacements, purchase of spare parts and components unavailability. To model the evolution of the cost function, it is necessary to model the simultaneous evolution of the fleet and of the stock. With that aim, the following variables are considered:

- I_t: states of the m components at time *t* (up or down),
- X_t: predicted times for future failures at time *t*,
- S_t: number of available spare parts at time *t*,
- D_t: predicted times for spare parts arrivals at time *t*,
- C_t: cumulated discounted cost at time *t*.

The process $(Y_t)_{t\geq 0} = (I_t, X_t, S_t, D_t, C_t, t)_{t\geq 0}$ is a PDMP which takes range in a space $E \times \mathbb{R}^m \times F \times \mathbb{R}^{p+2}$, where E and F are finite state spaces, m is the number of components and p is the number of pending orders. To simulate the evolution of $(Y_t)_{t\geq 0}$ by Monte-Carlo simulations, we consider the Markov chain $(Y_{T_n})_{n\geq 0} = (I_{T_n}, X_{T_n}, S_{T_n}, D_{T_n}, C_{T_n}, T_n)_{n\geq 0}$ with jump times $(T_n)_{n\geq 0}$. Between jumps, the evolution of $(Y_t)_{t\geq 0}$ is deterministic, and the whole process can be constructed from $(Y_{T_n})_{n\geq 0}$. Hence it is sufficient to simulate the evolution of the process at jump times, namely to simulate $(Z_n = Y_{T_n})_{n\geq 0}$. In our context, the jumps of the Markov chain

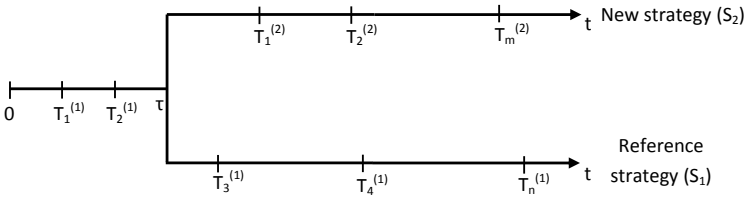

Fig. 2: Comparison of two strategies

correspond either to the failure of a component, to the arrival of a spare part, to an order of a new spare part or to a preventive replacement.

2.2 Assessment of the Net Present Value probabilistic distribution

Let t be the fixed horizon time. The NPV is defined as the difference of the cumulated random costs on $[0,t]$ between both strategies, with and without investments. The evolution of these strategies can be seen in figure 2, where τ is the first time where both strategies differ. For example, τ is the date of preventive replacement of some components in the new strategy. Hence the NPV is a function of two stochastic processes, which are dependent through their common history up to time τ. Before time τ, the strategies are identical and the NPV is null. After time τ, we assume that, given the common state of both processes at time τ^-, both strategies evolve independently. Hence, provided that $\tau < t$, the NPV on the time interval $[0,t]$ is identical to the NPV on $[\tau,t]$.

Let $C_t^{(i)}$ be the cumulative discounted cost up to time t for strategy $i \in \{1,2\}$. We need to estimate the probabilistic distribution of:

$$NPV\,(t) = C_t^{(1)} - C_t^{(2)}.$$

In the following, τ is assumed to be deterministic. As already told, for $t < \tau$, we have:

$$NPV\,(t) = 0$$

Therefore, we assume for the remainder of the document that $t \geq \tau$.
Let us denote by $F_{NPV(t)}$ the cumulative distribution function (c.d.f.) of the NPV, where

$$F_{NPV(t)}\,(u) = \mathbb{P}\left[NPV\,(t) \leq u\right],$$

all $u > 0$.
We have:

$$\begin{aligned} F_{NPV(t)}\,(u) &= \mathbb{P}\left[C_t^{(1)} - C_t^{(2)} \leq u\right] \\ &= \mathbb{E}\left[\Gamma_u\left(\left(Y_s^{(1)}, Y_w^{(2)}\right)_{0\leq s\leq t,\, 0\leq w\leq t-\tau}\right)\right] \\ &= \mathbb{E}\left[\Psi_u\left(\left(Z_n^{(1)}, Z_k^{(2)}\right)_{0\leq n\leq N_t^{(1)},\, 0\leq k\leq N_{t-\tau}^{(2)}}\right)\right] \end{aligned} \tag{1}$$

where Γ_u, Ψ_u are unknown functions, $\left(Y_s^{(i)}\right)_{s\geq 0}$ is the PDMP associated to strategy $i \in \{1,2\}$, $\left(Z_n^{(i)} = Y_{T_n^{(i)}}^{(i)}\right)_{0\leq n\leq N^{(i)}}$ is the underlying Markov chain and $N_t^{(i)}$ is the number of events occurred on $[0,t]$.
It can be seen in figure 2 that the strategy 2 is initialized at time τ. The strategy 2 is hence evaluated on a time interval with length $t-\tau$.
Now we present the different methods that we use to simulate these Markov chains.

3 Monte-Carlo and quasi Monte-Carlo methods

3.1 Monte-Carlo method

The Monte-Carlo method is widely used to solve equations and integrals when they cannot be solved analytically. In our context, we simulate independent stories of a fleet of components.
The estimation of the probabilistic distribution of the NPV amounts to estimating the integral of a function. This function, as shown in equation (1), depends on Markov chains which simulate the evolution of both strategies until horizon time *t*. For strategy $i \in \{1, 2\}$, the Markov chain is defined by:

$$\begin{cases} Z_0^{(i)} \sim f\left(V^{(i)}\right) \quad V^{(i)} \sim \mathcal{U}\left([0,1]^{m^{(i)}}\right) \\ Z_n^{(i)} = \phi_n\left(Z_{n-1}^{(i)}, U_n^{(i)}\right) \quad n \geq 1 \text{ et } U_n^{(i)} \sim \mathcal{U}\left([0,1]\right) \end{cases} \tag{2}$$

where $\sim \mathcal{U}\left([0,1]^p\right)$ means uniformly distributed on $[0,1]^p$ ($p \in \mathbb{N}^*$).

To simulate the evolution of the fleet of components by a Markov chain, we first need to simulate the first times to failure of the components. Then, $m^{(i)}$ represents the number of components for which first times to failure are simulated in strategy i. Once the first times to failure of all components are known, we simulate different possible events in each strategy. We assume that only one event occurs at the same time and hence we simulate random uniform variables on $[0, 1]$.

The c.d.f. of the NPV can be written as the integral of a function of uniform random variables. The number of random variables is the number of events occurring in both strategies up to time *t*. Hence it is also a random variable. Thus, the c.d.f. of the NPV can be written as:

$$F_{NPV(t)}(u) = \mathbb{E}\left[\Psi_{\phi,u}\left(f\left(V^{(1)}\right), U_1^{(1)}, U_2^{(1)}, \ldots, U_{N_t^{(1)}}^{(1)}, f\left(V^{(2)}\right), U_1^{(2)}, U_2^{(2)}, \ldots, U_{N_{t-\tau}^{(2)}}^{(2)}\right)\right] \tag{3}$$

where $N_t^{(i)}$ is the number of jumps for the Markov chain $\left(Z_n^{(i)}\right)_{n\geq 0}$ over $[0, t]$ for the strategy $i \in \{1, 2\}$ and the function $\Psi_{\phi,u}$ is unknown.
To assess the c.d.f. of the NPV by the MC method, we simulate sample paths of the Markov chains defined on equation (2) for each strategy up to time *t*. This provides a sample of NPVs. The c.d.f. of the NPV is next estimated by the empirical cumulative distribution function on this sample.
The main drawback of the MC method is that it provides random estimations and that its convergence speed is in $\frac{1}{\sqrt{N}}$. Thus, to obtain more accurate results, it is necessary to increase the number of simulations and therefore to increase the computational time. Within an optimization algorithm, the numerical assessment of the NPV distribution should be quick. Then, the MC method is not a relevant method because its computational time can be excessive. Quasi Monte-Carlo methods are well-known for often having a faster convergence than the MC method.

3.2 Quasi Monte-Carlo method

The quasi Monte-Carlo (QMC) method has been developed to improve the accuracy of estimations provided by the MC method. It substitutes uniform variables on $[0, 1]^d$ by deterministic sequences which have better uniformity properties. To measure the uniformity of a sequence, the discrepancy is used. Niederreiter in [9] defines the discrepancy as a quantitative measure for the deviation from uniform distribution. Thus, if the deviation is low or the discrepancy is low then the points are more uniformly distributed on $[0, 1]^d$. Sequences which have such good properties of uniformity are called Low Discrepancy Sequences (LDS). There are two families of low discrepancy sequences [6]: *lattices* and *digital nets/sequences*. For the last one, we have:

- the Van Der Corput sequence in dimension 1,
- the Halton sequence in dimension $d \geq 1$; it is a generalization of Van Der Corput sequences,
- the Faure sequence in dimension $d \geq 1$,
- the Sobol sequence in dimension $d \geq 1$.

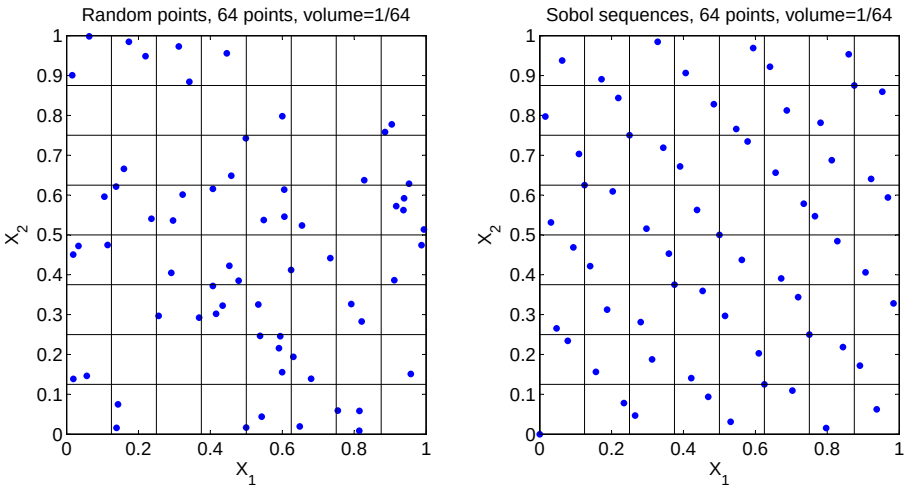

Fig. 3: Comparison of random uniform variable (left) and Sobol sequence (right)

We use the Sobol sequence for our numerical results. In figure 3, we compare a sample of 64 realizations of random uniform distribution on $[0,1]^2$ and 64 points of Sobol sequence. This figure illustrates the better uniformity of a Sobol sequence than a uniform random sample, for which many areas are left unexplored.

For the numerical assessment of the NPV c.d.f. at time *t* by the QMC method, the random uniform variables in equation (3) are substituted by terms of a LDS. Each element of the LDS is used to simulate one history of the fleet evolution and therefore one value of the NPV. The dimension of the LDS depends on the number of jumps of the Markov chains in both strategies. In the Matlab tool that we use, the dimension of the LDS must be known in advance and therefore it can not be a random variable. Thus to evaluate one value of the NPV, it is necessary to set the maximum number of jumps in both strategies, i.e. the maximum value of the random variables $N_t^{(1)}$ and $N_{t-\tau}^{(2)}$. We consequently choose $n^{(i)} \in \mathbb{N}^*$ such as $\mathbb{P}\left(n^{(i)} > m^{(i)} + N_t^{(i)}\right) > 1-\varepsilon, \varepsilon > 0$ and $i \in \{1,2\}$, and we build a LDS with dimension $n^{(1)} + n^{(2)}$. The function $\Psi_{\phi,u}$ in equation (3) is unknown and the Markov chain is simulated sequentially, just as for the MC method.

Setting *S* to be a $n^{(1)} + n^{(2)}$- dimensional LDS, we put $u_k^{(1)} = S_k\left(1 : n^{(1)}\right)$ to be the $n^{(1)}$ first components of the k^{th} element of the LDS *S* and $u_k^{(2)} = S_k\left(n^{(1)} + 1 : n^{(1)} + n^{(2)}\right)$, the $n^{(2)}$ last ones. The components of the sequence $u_k^{(i)}$ are used to simulate one history of the Markov chain $\left(Z_n^{(i)}\right)_{n\geq 0}$. The initialization of the Markov chain $\left(Z_n^{(i)}\right)_{n\geq 0}$ is done with the $m^{(i)}$ first components of $u_k^{(i)}$. As for the transitions, we use the next component of $u_k^{(i)}$ whenever we need a new random uniform variable on $[0,1]$.

The QMC method loses its effectiveness when the dimension of the LDS becomes large. Here the dimension of the LDS is linked to the number of jumps of the Markov chains. For the numerical assessment of investments plans, this number can become large. To overcome this limit, a suitable method for the simulation of Markov chain has been developed. It is called array quasi Monte-Carlo (AQMC).

3.3 Array quasi Monte-Carlo method

The array quasi Monte-Carlo (AQMC) method consists in simulating several copies of a Markov chain in parallel using elements of a LDS and introducing a negative dependence between the copies. The method has been introduced by Lécot and Tuffin [4] to simulate particles for solving a linear Boltzmann equation. At each time step of the simulation, the particles are reordered. Later, Lécot and Tuffin [5] and El Haddad, Lécot and L'Ecuyer [1] apply the method for the simulation of general Markov chains. The dependence is introduced by an ascending

sort function on copies at each step of the Markov chain. The goal is to improve the estimation of the theoretical distribution of the Markov chain at each step when compared to independent simulation of copies. Unlike the QMC method where an element of the LDS is used to simulate one path of the Markov chain, the AQMC method uses one element of the LDS to assess one step of the path. Thus the dimension of the LDS is not linked to the number of jumps of the Markov chain but is equal to the number of uniform random variables on $[0,1]$ needed to simulate one jump of the Markov chain. The AQMC method hence allows to reduce significantly the dimension of the LDS when compared to the QMC method.

To assess the distribution of NPV at the fixed horizon time by the AQMC method, we simulate *N* copies of the Markov chain in parallel for each strategy, defined in (2) and we reorder at each step. The associated costs with a strategy are evaluated at jump times of the Markov chain and the copies are sorted in ascending order on the jump times. In addition, the copies of both strategies are not sorted independently because their costs are dependent through their common history.
We now present the algorithm for evaluating the NPV by the AQMC method. First, we can see in equation (2) that the number of uniform random variables on $[0,1]$ necessary for the initialization of the Markov chains are different from the one required for the transitions. Then, two LDS are considered, say P_0 and P_1, with respective dimensions $m^{(1)}+m^{(2)}$ and 1. Elements of P_0 are used to initialize the Markov chain and elements of P_1 for the transitions. We use the following notations:

- *N* the number of copies in parallel,
- $P_0 = \{v_1, v_2, \dots, v_N\}$: *N* elements of the LDS with dimension $m^{(1)}+m^{(2)}$; $v_k\left(1:m^{(1)}\right)$ represents the $m^{(1)}$ first components of v_k and $v_k\left(m^{(1)}+1:m^{(1)}+m^{(2)}\right)$ the $m^{(2)}$ last ones,
- $P_1 = \{u_1, u_2, \dots\}$: elements of the LDS with dimension 1 and $\mathbf{u}_{l_1:l_2} = \{u_{l_1}, u_{l_1+1}, \dots, u_{l_2}\}$,
- $\mathbf{Z}_n^{(i)} = \left(Z_{n,1}^{(i)}, Z_{n,2}^{(i)}, \dots, Z_{n,N}^{(i)}\right)$: states of *N* copies in parallel for strategy $i \in \{1,2\}$ and $\mathbf{Z}_n^{(i)}(1:m)$ the *m* first states of $\mathbf{Z}_n^{(i)}$, $\mathbf{Z}_n^{(i)}(1:m) = \phi_n\left(\mathbf{Z}_{n-1}^{(i)}(1:m), \mathbf{u}_{l_1:(l_1+m-1)}\right)$ means that, for all $k \in \{1,2,\dots,m\}$, $Z_{n,k}^{(i)} = \phi_n\left(Z_{n-1,k}^{(i)}, u_{l_1+k-1}\right)$ where ϕ_n is defined by (2),
- $\mathbf{Z}_{(n)}^{(i)} = \left(Z_{n,(1)}^{(i)}, Z_{n,(2)}^{(i)}, \dots, Z_{n,(N)}^{(i)}\right)$: elements of $\mathbf{Z}_n^{(i)}$ sorted in ascending order according to jump times, namely: $Z_{n,(l_1)}^{(i)} \le Z_{n,(l_2)}^{(i)}$ if $T_{n,(l_1)}^{(i)} \le T_{n,(l_2)}^{(i)}\ \forall l_1, l_2$.

First part of the algorithm:

- *Initialization of strategy 1*
  - $\mathbf{Z}_0^{(1)} = \left(Z_{0,1}^{(1)}, Z_{0,2}^{(1)}, \dots, Z_{0,N}^{(1)}\right)$ with $Z_{0,k}^{(1)} = f\left(v_k\left(1:m^{(1)}\right)\right)$ for all $k \in \{1,2,\dots,N\}$,
  - Sort copies in ascending order according to jump times and relabel copies: $\mathbf{Z}_0^{(1)} \longleftarrow \mathbf{Z}_{(0)}^{(1)}$,
  - $m \longleftarrow \sum_{k=1}^{N} \mathbf{1}_{\{T_{0,k}^{(1)} \le \tau\}}$ the number of copies that have not reached τ yet,
  - $n \longleftarrow 1, l_1 \longleftarrow 1$,
- *While all copies have not reached τ, i.e. $m \neq 0$*
  - Simulation of the next step for the copies:
    $\mathbf{Z}_n^{(1)}(1:m) = \phi_n\left(\mathbf{Z}_{n-1}^{(1)}(1:m), \mathbf{u}_{l_1:l_1+m-1}\right)$ and $l_1 \longleftarrow l_1 + m$,
  - Sort copies in ascending order according to jump times and relabel copies: $\mathbf{Z}_n^{(1)} \longleftarrow \mathbf{Z}_{(n)}^{(1)}$
  - $m \longleftarrow \sum_{k=1}^{N} \mathbf{1}_{\{T_{n,k}^{(1)} \le \tau\}}$ the number of copies that have not reached τ yet,
  - $n \longleftarrow n+1$.

At the end of this first part, all copies of the Markov chain in strategy 1 have reached time τ. We next initialize the copies of the Markov chain in the strategy 2. The initialization of the strategy 2 depends on the state of $\left(\mathbf{Y}_t^{(1)}\right)_{t\ge 0}$ at time τ^-, and more specifically on the state $\mathbf{Z}_{N_\tau^{(1)}}^{(1)}$ of $\left(\mathbf{Y}_t^{(1)}\right)_{t\ge 0}$ at the jump time just before time τ. The second part of the algorithm now writes:

- *Initialization of the strategy 2*
  - $\mathbf{Z}_0^{(2)} = \left(Z_{0,1}^{(2)}, Z_{0,2}^{(2)}, \ldots, Z_{0,N}^{(2)}\right)$ with $Z_{0,k}^{(2)} = f_{Z^{(1)}_{N_\tau^{(1)},k}}\left(v_k\left(m^{(1)}+1 : m^{(1)}+m^{(2)}\right)\right)$ for all $k \in \{1,2,\ldots,N\}$.

At this step, both strategies have been initiated. We construct a coupled chain $\mathbf{Z}$, which keeps together the copies in both strategies which have the common history. Then, we have $\mathbf{Z} = \left(\mathbf{Z}^{(1)}_{N_\tau^{(1)}}, \mathbf{Z}_0^{(2)}\right) = \left(\left(Z^{(1)}_{N_\tau^{(1)},1}, Z^{(2)}_{0,1}\right), \ldots, \left(Z^{(1)}_{N_\tau^{(1)},N}, Z^{(2)}_{0,N}\right)\right)$. Next, we simulate the copies of the strategy 1 in a coupled chain $\mathbf{Z}$ following the first part of the above algorithm where τ is now substituted by the horizon time *t*. Thus, we sort in ascending order the elements of the coupled chain in relation to jump times in strategy 1. The next steps of copies of strategy 1 are simulated by using the next elements of the LDS P_1. When all copies in parallel of the strategy 1 have reached the horizon time *t*, we simulate copies of the strategy 2 by using the next elements of the LDS P_1. Thus, elements of the coupled chain are sorted in ascending order in relation to jump times in the strategy 2. Both strategies can be simulated one after another because they evolve independently from time τ (given their common past up to time τ). When all copies in both strategies have reached the horizon time *t*, the NPV are estimated by making the difference between the cumulative discounted costs in both strategies.

3.4 Randomization methods

QMC and AQMC methods do not provide confidence intervals for the estimations. To evaluate these intervals, we use randomized version of LDS. Randomized version of LDS are random sequences which have the two following properties [7]:

- each randomized point is uniformly distributed on $[0,1]^d$,
- the regularity of the point is preserved (in the sense of low discrepancy).

Several randomizations methods are available [6]: *random shift*, *digital nets*, *scrambling*, ... In this paper, we use the *random shift* method.

4 Numerical results on a fictitious case study

A fleet of *m* identical and independent components is considered, which share a common stock of spare parts. The lifetime of component have a Weibull distribution with parameters $\lambda = 60$ and $\beta = 3$. The components are assumed to be new at time $t = 0$ and the operation horizon time is 60 years.

Definition 4.1 *The cumulative distribution function for the Weibull distribution with parameters λ and β is given by:*

$$F(x) = \left(1 - e^{-\left(\frac{x}{\lambda}\right)^\beta}\right) \times \mathbf{1}_{\{x \geq 0\}}$$

4.1 Strategies

We compare the followings strategies:

- Strategy 1 (corrective maintenance):
  A failed component is replaced by a new one if a spare part is available. If the stock is empty at failure time, the component becomes unavailable. It is next replaced as soon as a spare part becomes available.
- Strategy 2 (preventive maintenance):
  In addition to corrective replacements, a complete overhaul takes place at time $\tau = 20$, involving the replacement of all components with new ones.

4.2 Stock logistic

At the initial time, the stock has one spare part. The stock is used only for corrective replacements, so it is a safety stock. An order for a new spare part is made as soon as a component fails. The payment for a new spare part is cashed on delivery. The supply time is one year.

4.3 Economics values

A continuous discount rate is taken into account with $\alpha = 7.5\%$ and costs are discounted at time 0: if C is some cost at time *t*, then the discounted cost is $C \times e^{-\alpha t}$. The cost of a corrective replacement is $c_r = 190$ (not including the price of spare part) and the daily cost of downtime is $c_{ind} = 160$. The preventive replacements charged $c_p = 190$ without the purchasing price of spare parts. The purchasing price of one spare-part is $c_A = 500$.

4.4 Comparison of methods for the relative error

We first compare both deterministic methods (QMC and AQMC) for their relative errors on the expected NPV and on the NPV probabilistic distribution at time *t*. After, we compare the MC method to the randomized versions (RQMC and RAQMC) with the means of their relative errors on the expected NPV and on the NPV distribution.

4.4.1 Relative error on the expected NPV

Let us denote by $\mu = \mathbb{E}\left[NPV\left(t\right)\right]$ the expected NPV at time *t* and $\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} NPV_i\left(t\right)$ the estimation of μ by (A)QMC methods. The relative error on the expected NVP is defined by:

$$\varepsilon = \left|\frac{\hat{\mu} - \mu}{\mu}\right|$$

To compare the random methods (MC, RQMC and RAQMC), we evaluate the mean of the relative error on *J* independent simulations (resp. randomizations) of the MC (resp. (A)QMC) method. We have $\varepsilon_j = \left|\frac{\hat{\mu}_j - \mu}{\mu}\right|$ where $\hat{\mu}_j = \frac{1}{N}\sum_{i=1}^{N} NPV_i^j\left(t\right)$ is the estimation of μ obtained in the j^{th} simulation of the MC method or randomization of (A)QMC methods. Then, the mean of the relative error on the expected NPV is:

$$\overline{\varepsilon} = \frac{1}{J}\sum_{j=1}^{J} \varepsilon_j$$

4.4.2 Relative error on the $NPV\left(t\right)$ probabilistic distribution

We consider a set $I = \{a_1, a_2, \ldots, a_{n_I}\}$ of n_I possible values of $NPV\left(t\right)$ such as $a_1 < a_2 < \cdots < a_{n_I}$ and $\mathbb{P}\left(NPV(t) \leq a_{n_I}\right) - \mathbb{P}\left(NPV(t) \leq a_1\right)$ is close to 95%. We put $\epsilon_{a_j} = \left|\frac{\widehat{P}_{NPV(t)}^{a_j} - P_{NPV(t)}^{a_j}}{P_{NPV(t)}^{a_j}}\right|$ where $P_{NPV(t)}^{a_j} = \mathbb{P}\left(NPV(t) \leq a_j\right)$ and $\widehat{P}_{NPV(t)}^{a_j} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{NPV_i(t) \leq a_j\}}$ is the $P_{NPV(t)}^{a_j}$ estimation with (A)QMC methods. ϵ_{a_j} represents the relative error at point a_j on the NPV c.d.f.. We define the relative error on the $NPV\left(t\right)$ probabilistic distribution by:

$$\mathcal{E} = \frac{1}{n_I}\sum_{j=1}^{n_I} \epsilon_{a_j}$$

To evaluate the average error on the $NPV\left(t\right)$ probabilistic distribution by R(A)QMC methods and the MC method, we perform *J* independent simulations of the MC method or randomizations of (A)QMC methods and we obtain:

$$\overline{\mathcal{E}} = \frac{1}{J}\sum_{j=1}^{J} \mathcal{E}_j$$

where $\mathcal{E}_j$ is the relative error on the $NPV\left(t\right)$ obtained in the j^{th} simulation of the MC method or randomization of (A)QMC methods.

4.5 Four components

We consider four identical and independent components which share a common stock of spare parts. To obtain reference values, we consider $N = 10^8$ independent stories in the MC method. We can see in figure 4 that

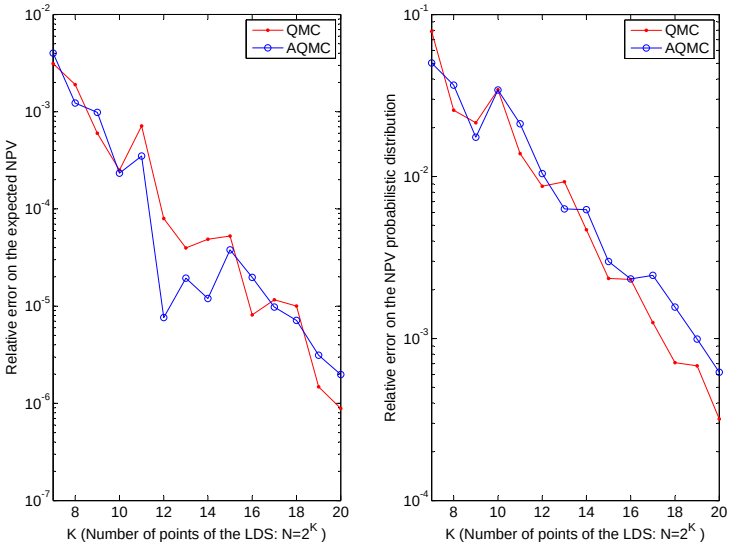

Fig. 4: Relative error on the expected NPV and the NPV probabilistic distribution at 60 years - 4 components

(A)QMC methods have a good estimation for the expected NPV and for the probabilistic distribution of the NPV. As expected, the relative error on the NPV distribution converges towards 0 when the number of points increases. Moreover the relative error on the expected NPV is lower than 10^{-4} from $N = 2^{12} = 4096$. In figure 5, we compare randomized methods (MC, RQMC and RAQMC) obtained with $J = 500$ independent simulations or randomizations. In this figure, we observe that all methods have a good evaluation for the expected NPV and the probabilistic distribution of the NPV. We can see that the MC method needs $N = 2^{14} = 16384$ simulations to obtain a relative error on the expected NPV around 10^{-3} whereas R(A)QMC methods require only $N = 2^8 = 256$ points of LDS. Then (A)QMC methods have a reduction factor equal to $2^6 = 64$ points. Moreover, we can see that the slope of R(A)QMC methods is greater than the MC method one. Thus, the reduction factor increases as the number of points (N) increases. Furthermore, the relative error on the NPV probabilistic distribution of R(A)QMC methods is also lower than the MC method one.

5 Conclusion

In this paper, we have proposed to use quasi Monte-Carlo methods for the assessment of the Net Present Value distribution of an investments plan. In general, we have observed a good estimation of the NPV distribution by QMC and AQMC methods. Moreover, in our example, these methods provide a clear efficiency gain when compared with the Monte-Carlo method. More specifically, for a given relative error, quasi Monte-Carlo methods require a smaller amount of points than MC method, so that accurate results are obtained faster than with the MC method. This point is crucial for the use within an optimization algorithm, and the numerical optimization of investments plans should be faster with (A)QMC methods than with the MC method.

As for the comparison between QMC and AQMC methods, we have seen that, when the number of components of the fleet increases, the dimension of the LDS in the QMC method increases. Previous studies [2] show that the QMC method loses its efficiency when the dimension of the LDS becomes large. Thus, the QMC method does not seem to be suitable for a large fleet of components. The AQMC method does not suffer from the same drawback and the dimension of the involved LDS does not vary when the number of components increases. Therefore, it seems to be suitable for a large fleet of components.

The next step of this study is to introduce quasi Monte-Carlo methods within an algorithm optimization and to evaluate the convergence speeds for the different methods.

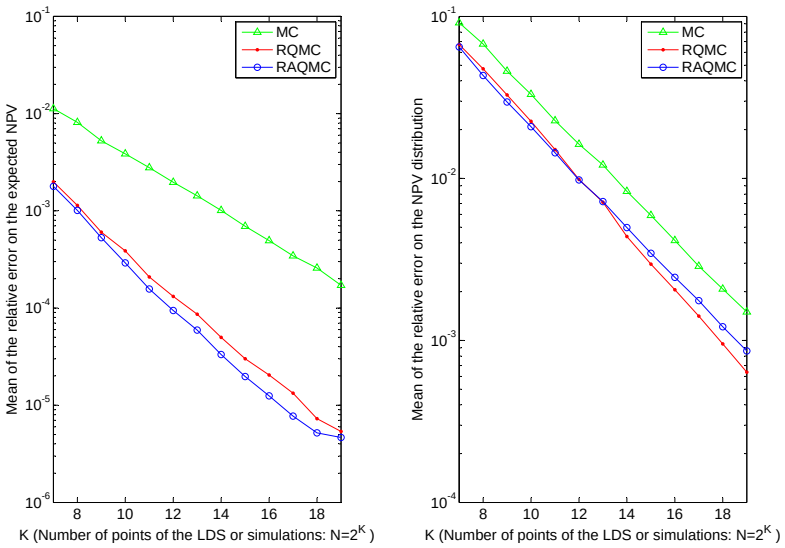

Fig. 5: Mean of the relative error on the expected NPV and the NPV probabilistic distribution at 60 years - 4 components

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