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MONOTONE MARKOV PROCESSES WITH RESPECT TO THE REVERSED HAZARD RATE ORDERING: AN APPLICATION TO RELIABILITY

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Abstract

We consider a repairable system with a finite state space which evolves in time according to a Markov process as long as it is working. We assume that this system is getting worse and worse while running: if the up-states are ranked according to their degree of increasing degradation, this is expressed by the fact that the Markov process is assumed to be monotone with respect to the reversed hazard rate and to have an upper triangular generator. We study this kind of process and apply the results to derive some properties of the stationary availability of the system. Namely, we show that, if the duration of the repair is independent of its completeness degree, then the more complete the repair, the higher the stationary availability, where the completeness degree of the repair is measured with the reversed hazard rate ordering.

Keywords: Reliability; reversed hazard rate ordering; monotone Markov processes; stationary availability

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1. Introduction

Let us consider a repairable system such that different completeness degrees are possible for the repair, that go from a 'minimal' up to a 'complete' repair. Then, a natural problem (and it is of great importance in industry) is to look for the optimal degree of the repair, that is, find the degree which optimizes a certain criterion. Here, we concentrate on the complete repair and we want to give conditions under which it is optimal. To measure the performance of the system, we use the stationary (or long-run) availability, that is, the probability for the system to be up when in steady state. Then, under which conditions is a complete repair optimal?

We may first propose an intuitive answer: if the system gets 'worse and worse' when running and if it takes the same time to achieve a complete or a minimal repair, then the stationary availability should be higher as the repair is complete. In other words, for a system with some kind of 'ageing' property and such that the duration of the repair is independent of its completeness degree, a complete repair should be optimal. The point indeed is to find the right ageing notion for our study. Namely, under which kind of ageing property is a complete repair optimal?

Before answering such a question, we must specify the model for our system. We assume that it behaves according to a Markov process with a finite state space up to its first failure,

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and in the same way after any repair. It is subject to different kinds of failure. To each corresponds a repair with a random duration and a general distribution. The duration of the repair is independent of its completeness degree. After any repair, the new start of the system is independent of the previous evolution of the system and is controlled by a fixed distribution on the up-states.

We now come to the mathematical translation of the increasing degradation of the system when running, and we translate this property with some stochastic monotonicity for the underlying Markov process. Such properties have been much studied in the recent literature; see, for instance, Brown and Chaganty (1983), Shaked and Shanthikumar (1987), (1988), Shanthikumar (1988), Karasu and Özekici (1989), Li and Shaked (1995), (1997) or Kijima (1997), (1998), where different stochastic monotonicities are considered. The problem then is to find the one most adapted to our study. The 'usual' stochastic ordering has been, up to now, the most commonly used in reliability (we may think of Barlow and Proschan (1975), of course, but it is still the case in more recent books devoted to various stochastic orderings such as Stoyan (1983), Shaked and Shanthikumar (1994) or Szekli (1995) in their applications to reliability). Then, a natural question is to wonder whether it is adapted to our study. The answer is negative, and we show that in order for the complete repairs to be optimal, the underlying Markov process needs to be monotone with respect to a stronger ordering than the usual stochastic one: it needs to be monotone with respect to the reversed hazard rate ordering. This kind of monotone process has been recently studied by Kijima (1998) (in a more general context than ours) and our own paper is mainly based on his work. Actually, because of a technical point, we have to limit ourselves to a smaller class of processes than those studied by Kijima and we concentrate on those with an upper triangular generator. Note that this restriction may also be motivated by the fact that, with such a generator, we show that the state of the system at time s is greater than at time t $(0 \leq t \leq s)$ in the sense of the reversed hazard rate ordering. This appears to be the mathematical translation of the 'increasing degradation of the system when running' we were looking for. Then, for a system with such a behaviour, we show that a complete repair is optimal or, more generally, that the stationary availability is all the higher as the repair is complete. Here again, the degree of completeness of the repair is measured with the reversed hazard rate ordering and the usual stochastic ordering is inadequate.

If we compare this with what may be found in the literature, we notice that some other examples of optimization problems where the optimum is attained under reversed hazard rate conditions, instead of just simple usual stochastic order conditions, may be found in Shanthikumar *et al.* (1991) and Cheng and Righter (1995) (both in queueing systems) or in Eeckhoudt and Gollier (1995) (in risk theory).

This paper is subdivided as follows. In the next section, we recall and complete known results on monotone Markov processes with respect to the reversed hazard rate ordering. Section 3 is devoted to our application to reliability.

Throughout the paper, 'increasing' and 'decreasing' mean, respectively, 'non-decreasing' and 'non-increasing'.

2. Some results on reversed hazard rate monotone Markov processes with upper triangular generators

We first summarize a few well-known facts about the reversed hazard rate ordering (see Keilson and Sumita (1982), Shaked and Shanthikumar (1994), Kijima (1997) or Block *et al.* (1998) for instance).

2.1. The reversed hazard rate ordering

Let $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ be two probability row vectors on $\{1, \dots, m\}$. We recall that $\boldsymbol{\nu}_1$ is said to be greater than $\boldsymbol{\nu}_2$ in the sense of reversed hazard rate ordering, denoted by $\boldsymbol{\nu}_1 \prec_{\mathrm{rh}} \boldsymbol{\nu}_2$, if and only if

$$\left(\sum_{k=1}^{j} \boldsymbol{\nu}_1(k)\right)\left(\sum_{k=1}^{i} \boldsymbol{\nu}_2(k)\right) \le \left(\sum_{k=1}^{i} \boldsymbol{\nu}_1(k)\right)\left(\sum_{k=1}^{j} \boldsymbol{\nu}_2(k)\right), \quad \text{for any } 1 \le i \le j \le m, \tag{2.1}$$

which may also be written as

$$\frac{\sum_{k=1}^{j} \boldsymbol{\nu}_1(k)}{\sum_{k=1}^{i} \boldsymbol{\nu}_1(k)} \le \frac{\sum_{k=1}^{j} \boldsymbol{\nu}_2(k)}{\sum_{k=1}^{i} \boldsymbol{\nu}_2(k)}, \quad \text{for any } 1 \le i \le j \le m, \tag{2.2}$$

when defined, using the convention $0/0 = 0$.

Also, it is convenient to note that inequalities (2.1) or (2.2) are required only for $1 \le i \le m-1$ and $j = i + 1$ to get $\boldsymbol{\nu}_1 \prec_{\mathrm{rh}} \boldsymbol{\nu}_2$ (Keilson and Kester (1977)).

Another way to express the reversed hazard rate ordering is to introduce the upper triangular $m \times m$ matrix $\boldsymbol{U}$ such that every non-zero element is equal to 1 (Keilson and Sumita (1982)). The matrix $\boldsymbol{U}$ is non-singular and $\boldsymbol{U}^{-1}$ is the upper triangular matrix such that the only non-zero elements are $\boldsymbol{U}^{-1}_{i,i} = 1$ for $1 \le i \le m$ and $\boldsymbol{U}^{-1}_{i,i+1} = -1$ for $1 \le i \le m-1$. Then, $\boldsymbol{\nu}_1 \prec_{\mathrm{rh}} \boldsymbol{\nu}_2$ is equivalent to

$$\begin{pmatrix} \boldsymbol{\nu}_1 \\ \boldsymbol{\nu}_2 \end{pmatrix} \boldsymbol{U} \in \mathrm{TP}_2.$$

(Let us recall that a matrix is said to be TP_2 (totally positive of order 2) if and only if each of its minors of order 2 is non-negative; see Karlin (1968) for details.)

We now recall an important result in the study of reversed hazard rate monotone Markov processes (Kijima (1997, Corollary 3.3)).

Lemma 2.1. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two non-negative matrices such that $\boldsymbol{A}$ has m columns, $\boldsymbol{B}$ is a $m \times m$ matrix, $\boldsymbol{AU} \in \mathrm{TP}_2$ and $\boldsymbol{BU} \in \mathrm{TP}_2$. If $\boldsymbol{U}^{-1}\boldsymbol{BU} \ge 0$, then $\boldsymbol{ABU} \in \mathrm{TP}_2$.*

In case $\boldsymbol{B}$ is a stochastic $m \times m$ matrix, we can easily check that $\boldsymbol{BU} \in \mathrm{TP}_2$ is now equivalent to $\boldsymbol{B}_{i,\bullet} \prec_{\mathrm{rh}} \boldsymbol{B}_{i+1,\bullet}$ and that $\boldsymbol{U}^{-1}\boldsymbol{BU} \ge 0$ is equivalent to $\boldsymbol{B}_{i,\bullet} \prec_{\mathrm{st}} \boldsymbol{B}_{i+1,\bullet}$ for any $1 \le i \le m-1$ (Kijima (1997, Corollary 3.5)), where $\boldsymbol{B}_{i,\bullet}$ is the ith row of $\boldsymbol{B}$. As the reversed hazard rate ordering implies the usual stochastic ordering, $\boldsymbol{BU} \in \mathrm{TP}_2$ now implies that $\boldsymbol{U}^{-1}\boldsymbol{BU} \ge 0$. We derive the following result:

Corollary 2.1. *Let $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ be two probability row vectors on $\{1, \dots, m\}$ such that $\boldsymbol{\nu}_1 \prec_{\mathrm{rh}} \boldsymbol{\nu}_2$ and let $\boldsymbol{B}$ be a stochastic $m \times m$ matrix such that $\boldsymbol{B}_{i,\bullet} \prec_{\mathrm{rh}} \boldsymbol{B}_{i+1,\bullet}$ for any $1 \le i \le m-1$. Then $\boldsymbol{\nu}_1\boldsymbol{B} \prec_{\mathrm{rh}} \boldsymbol{\nu}_2\boldsymbol{B}$.*

Finally, we give a complementary result that may be found in Joag-dev *et al.* (1995) (see Theorem 2.1, or more precisely the following remark) with a different formulation.

Lemma 2.2. *Let $\boldsymbol{\xi}^1$ and $\boldsymbol{\xi}^2$ be two non-negative row vectors on $\{1, \dots, m\}$ (non-identically null) such that*

$$\begin{pmatrix} \boldsymbol{\xi}^1 \\ \boldsymbol{\xi}^2 \end{pmatrix} \boldsymbol{U} \in \mathrm{TP}_2.$$

Let z and w be two $m \times 1$ column vectors such that w is positive and decreasing componentwise and

$$\left(\frac{z_i}{w_i}\right)_{1 \le i \le m}$$

is increasing componentwise. Then

$$\frac{\xi^1 z}{\xi^1 w} \le \frac{\xi^2 z}{\xi^2 w}.$$

2.2. Reversed hazard rate monotone Markov processes with upper triangular generators

Let (X_t) be a Markov process on the finite state space $\{1, \dots, m+1\}$ and $(P_t(i, j))_{1 \le i, j \le m+1}$ be its associated transition kernel: $P_t(i, j) = \mathrm{P}_i(X_t = j)$ for any $1 \le i, j \le m+1$ and $t \ge 0$, where $\mathrm{P}_i(\cdot)$ is the conditional distribution $\mathrm{P}_i(\cdot) = \mathrm{P}(\cdot \mid X_0 = i)$.

For such a Markov process, let $A = (a_{i,j})_{1 \le i, j \le m+1}$ be its (infinitesimal) generator. We recall that (see Anderson (1991) for details):

$$a_{i,j} = \lim_{t \to 0^+} \frac{P_t(i, j)}{t}, \quad \text{for any } 1 \le i,\ j \le m+1 \text{ such that } i \ne j,$$

$$a_{i,i} = -\lim_{t \to 0^+} \frac{1 - P_t(i, j)}{t} = -\sum_{j \ne i} a_{i,j}, \quad \text{for any } 1 \le i \le m+1.$$

Also, for $1 \le i \le m+1$, let $P_t(i, \bullet)$ be the ith row of $(P_t(i, j))_{1 \le i, j \le m+1}$.

Following Kijima, let us now recall the definition of a monotone Markov process with respect to the reversed hazard rate ordering (reversed hazard rate monotone Markov process for short) and its characterization in terms of its generator.

Definition 2.1. We say that (X_t) is a reversed hazard rate monotone Markov process (and we write $(X_t) \in \mathcal{M}_{\mathrm{rh}}$) if and only if $P_t(i, \bullet) \prec_{\mathrm{rh}} P_t(i+1, \bullet)$ for any $1 \le i \le m,\ t \ge 0$ (or equivalently $P_t U \in \mathrm{TP}_2$).

Proposition 2.1. (Kijima (1998).) *The Markov process (X_t) is reversed hazard rate monotone if and only if*

$$a_{i,j} = 0 \quad \textit{for any } 1 \le j \le i-2 \le m+1,$$
$$a_{i,j} \le a_{i+1,j}, \quad \textit{for any } 3 \le i+2 \le j \le m+1.$$

In the same paper, Kijima also showed (in a more general context than ours) that for a reversed hazard rate monotone Markov process, the relation $P_t(1, \bullet) \prec_{\mathrm{rh}} P_s(1, \bullet)$ is valid for any $0 \le t \le s$. If we limit ourselves to upper triangular generators, then any state i plays the same role as state 1 in Kijima's work. This explains the *reverse* direction of the following equivalence.

Proposition 2.2. *If (X_t) is a reversed hazard rate monotone Markov process, then*

$$(P_t(i, \bullet) \prec_{\mathrm{rh}} P_s(i, \bullet) \textit{ for any } 1 \le i \le m+1,\ 0 \le t \le s)$$
$$\iff (A \textit{ is upper triangular}). \tag{2.3}$$

Proof. Let us assume that A is upper triangular and let $1 \le i \le m+1,\ 0 \le t \le s$.

Let us first check that $P_0(i, \bullet) \prec_{\mathrm{rh}} P_{s-t}(i, \bullet)$ or, equivalently,

$$\left(\sum_{k=1}^{j+1} P_0(i, k)\right)\left(\sum_{k=1}^{j} P_{s-t}(i, k)\right) \le \left(\sum_{k=1}^{j} P_0(i, k)\right)\left(\sum_{k=1}^{j+1} P_{s-t}(i, k)\right),$$

for any $1 \le j \le m$. This inequality may also be written as

$$\mathbf{1}_{\{i\le j+1\}}\Big(\sum_{k=1}^{j} \boldsymbol{P}_{s-t}(i,k)\Big) \le \mathbf{1}_{\{i\le j\}}\Big(\sum_{k=1}^{j+1} \boldsymbol{P}_{s-t}(i,k)\Big), \quad \text{for any } 1 \le j \le m. \tag{2.4}$$

For $j+1 < i$ or $j \ge i$, it is clear. For $j = i-1$ (and $i \ge 2$), as $\boldsymbol{P}_t$ is upper triangular (just as $\boldsymbol{A}$ is), we have $\sum_{k=1}^{i-1} \boldsymbol{P}_{s-t}(i,k) = 0$ and (2.4) is again clear. We now know that $\boldsymbol{P}_0(i,\bullet) \prec_{\text{rh}} \boldsymbol{P}_{s-t}(i,\bullet)$ for any $1 \le i \le m+1,\ 0 \le t \le s$.

As $(X_t) \in \mathcal{M}_{\text{rh}}$, we also have $\boldsymbol{P}_t(k,\bullet) \prec_{\text{rh}} \boldsymbol{P}_t(k+1,\bullet)$ for any $1 \le k \le m,\ t \ge 0$. Then, Corollary 2.1 (with $m+1$ substituted for m) implies that

$$\sum_{k=1}^{m+1} \boldsymbol{P}_0(i,k)\boldsymbol{P}_t(k,\bullet) = \boldsymbol{P}_t(i,\bullet) \prec_{\text{rh}} \sum_{k=1}^{m+1} \boldsymbol{P}_{s-t}(i,k)\boldsymbol{P}_t(k,\bullet) = P_s(i,\bullet).$$

The reverse direction may be proved in a similar way to that in Kijima (1998, Lemma 3.1(iii)).

Remark 2.1. It is easy to check that the same result is still valid for a monotone Markov process with respect to the usual stochastic ordering, where the usual stochastic ordering is substituted for the reversed hazard rate ordering in (2.3).

We now concentrate on those reversed hazard rate monotone Markov processes with upper triangular generators and we use the following notation.

Definition 2.2. We say that $(X_t) \in \mathcal{M}_{\text{rh}}^{\text{U}}$ if and only if $(X_t) \in \mathcal{M}_{\text{rh}}$ and if its generator is upper triangular with non-zero diagonal coefficients, except for the last one which is null.

Note that for an upper triangular generator, the last diagonal coefficient is necessarily null, so that this is not an assumption. (The matrix $\boldsymbol{A}$ is a special case of what is called in the literature a *lossy* generator; see Kijima (1997) for instance.) The other diagonal coefficients are assumed to be non-zero, which ensures that $\{1,\dots,m\}$ is a non-absorbing set. This allows us to avoid technical discussions which have no object for our application to reliability.

We now use the following notation: for any $1 \le i, l \le m$, let T_i^{l+1} be the hitting-time of $\{l+1,\dots,m\}$ for the process (X_t) starting from state i ($T_i^{l+1} = \inf(t \ge 0 \mid X_t > l, X_0 = i)$) and let $\tau_i^{l+1}(t)$ be the associated hazard rate.

Proposition 2.3. *For* $(X_t) \in \mathcal{M}_{\text{rh}}^{\text{U}}$*:*

(i) $\tau_i^{l+1}(t)$ *is increasing in* t*, i.e.,* T_i^{l+1} *is of increasing hazard rate, for* $1 \le i \le l \le m$.

(ii) $\tau_{i+1}^{l+1}(t) \ge \tau_i^{l+1}(t)$ *for* $1 \le i \le l-1 \le m$ *and* $t \ge 0$.

(iii) $\tau_i^{l}(t) \ge \tau_i^{l+1}(t)$ *for* $1 \le i \le l-1 \le m$ *and* $t \ge 0$.

Proof. The first point is similar to Theorem 4.2 of Kijima (1998). Indeed, in that paper, Kijima showed that for a reversed hazard rate monotone process (X_t), T_1^{l+1} is of increasing hazard rate. With an upper triangular generator, we get the same result when the process starts from any $1 \le i \le m$ just as in Proposition 2.2. Then, we need only prove (ii) and (iii). To that end, we use the same formulation for $\tau_i^{l+1}(t)$ as Kijima for his Theorem 4.2: let $1 \le i \le l \le m$ and let $\bar{F}_i^{l+1}$ be the survival function of T_i^{l+1}. We find that

$$\tau_i^{l+1}(t) = \frac{-(\bar{F}_i^{l+1})'(t)}{\bar{F}_i^{l+1}(t)} = \frac{\sum_{j=1}^{l} \boldsymbol{P}_t(i,j)\sum_{k=l+1}^{m+1} a_{j,k}}{\sum_{j=1}^{l} \boldsymbol{P}_t(i,j)}.$$

Using the following equality

$$\sum_{j=1}^{l} a'_j b_j = \sum_{j=1}^{l-1} (a'_j - a'_{j+1}) \sum_{n=1}^{j} b_n + a'_l \sum_{n=1}^{l} b_n, \quad \text{for } l \geq 2$$

with

$$a'_j = \sum_{k=l+1}^{m+1} a_{j,k} \quad \text{and} \quad b_j = \frac{P_t(i,j)}{\sum_{n=1}^{l} P_t(i,n)}$$

(note that $\sum_{n=1}^{l} b_n = 1$), we find that

$$\tau_i^{l+1}(t) = \sum_{j=1}^{l-1} \left(\sum_{k=l+1}^{m+1} a_{j,k} - \sum_{k=l+1}^{m+1} a_{j+1,k} \right) \frac{\sum_{n=1}^{j} P_t(i,n)}{\sum_{n=1}^{l} P_t(i,n)} + \sum_{k=l+1}^{m+1} a_{l,k} \times 1. \tag{2.5}$$

Using the facts that

$$\sum_{k=l+1}^{m+1} a_{j,k} - \sum_{k=l+1}^{m+1} a_{j+1,k} \leq 0 \quad \text{for any } j \leq l-1,$$

$$\frac{\sum_{n=1}^{j} P_t(i,n)}{\sum_{n=1}^{l} P_t(i,n)} \geq \frac{\sum_{n=1}^{j} P_t(i+1,n)}{\sum_{n=1}^{l} P_t(i+1,n)} \quad \text{for any } 1 \leq i \leq l-1 \leq m$$

(both because $(X_t) \in \mathcal{M}_{\mathrm{rh}}^{\mathrm{U}}$), we now derive from (2.5) that $\tau_i^{l+1}(t) \leq \tau_{i+1}^{l+1}(t)$ for any $1 \leq i \leq l-1 \leq m$, so (ii) is proved.

The third point easily follows from the fact that $(\bar{F}_i^{l+1}/\bar{F}_i^{l})(t)$ is increasing in t, due to the inequality

$$\frac{\bar{F}_i^{l+1}(t)}{\bar{F}_i^{l}(t)} = \frac{\sum_{j=1}^{l} P_t(i,j)}{\sum_{j=1}^{l-1} P_t(i,j)} \leq \frac{\sum_{j=1}^{l} P_s(i,j)}{\sum_{j=1}^{l-1} P_s(i,j)} = \frac{\bar{F}_i^{l+1}(s)}{\bar{F}_i^{l}(s)}, \quad \text{for } t \leq s$$

(see Proposition 2.2). Writing $(\bar{F}_i^{l+1}/\bar{F}_i^{l})' \geq 0$, we conclude that $\tau_i^{l}(t) \geq \tau_i^{l+1}(t)$, which completes the proof.

We now introduce the $m \times m$ matrix G such that $G_{i,j} = \int_0^{+\infty} P_t(i,j)\,\mathrm{d}t$ for any $1 \leq i, j \leq m$. The value $G_{i,j}$ represents the mean duration spent in state j when the process starts from state i and will be of great importance in our application to reliability. Note that, for $(X_t) \in \mathcal{M}_{\mathrm{rh}}^{\mathrm{U}}$, each $G_{i,j}$ is finite and that $G = -A_1^{-1}$, where A_1 is the *north-west* $m \times m$ *truncation* of A (see Kijima (1997, Theorem 4.25) for instance).

The next result is the key to our application to reliability.

Proposition 2.4. *If* $(X_t) \in \mathcal{M}_{\mathrm{rh}}^{\mathrm{U}}$, *then* $U^{-1}GU \geq 0$ *and* $GU \in \mathrm{TP}_2$.

Proof. Let $(X_t) \in \mathcal{M}_{\mathrm{rh}}^{\mathrm{U}}$. Then we know that $P_t U \in \mathrm{TP}_2$ (because $(X_t) \in \mathcal{M}_{\mathrm{rh}}$; see Definition 2.1), so that $U^{-1} P_t U \geq 0$ for any $t \geq 0$ (see the few lines following Lemma 2.1). With an integration, we easily derive $U^{-1}GU \geq 0$, which is the required inequality.

We now have to prove that $GU \in \mathrm{TP}_2$. Let $E = GU$, that is, let $E = (E_{i,j})_{1 \leq i,j \leq m}$ with $E_{i,j} = \sum_{k=1}^{j} G_{i,k}$, for any $1 \leq i, j \leq m$. The matrix E clearly is upper triangular, as A and

$G(= -A_1^{-1})$ are. Also, E is positive componentwise. For $1 \leq i \leq m-1$, we have to prove that

$$E_{i,j+1}E_{i+1,j} \leq E_{i,j}E_{i+1,j+1}, \tag{2.6}$$

for any $1 \leq j \leq m-1$.

We call this property (P_i); let us show (P_i) by decreasing induction.

It is clear that (P_{m-1}) holds, because $E_{m,j} = 0$ for any $1 \leq j \leq m-1$ (E is upper triangular).

Now let $1 \leq i \leq m-2$. Let us assume that (P_k) is true for $i+1 \leq k \leq m-1$ and let us show that (P_i) is true. If $i \geq j$, then $E_{i+1,j} = 0$ and (2.6) is true. Now let $j \geq i+1$. As $E = GU$ and $G = -A_1^{-1}$, we have $A_1 E = -U$, which implies that

$$\sum_{k=1}^{m} a_{i,k}E_{k,j} = \sum_{k=i}^{j} a_{i,k}E_{k,j} = -\mathbf{1}_{\{i\leq j\}} = -1.$$

As A is upper triangular, we find that

$$a_{i,i}E_{i,j} = -\sum_{k=i+1}^{j} a_{i,k}E_{k,j} - 1 = -\sum_{k=i+1}^{j+1} a_{i,k}E_{k,j} - 1 \tag{2.7}$$

(because $E_{j+1,j} = 0$) and

$$a_{i,i}E_{i,j+1} = -\sum_{k=i+1}^{j+1} a_{i,k}E_{k,j+1} - 1 \tag{2.8}$$

in the same way.

As $a_{i,i} < 0$, inequality (2.6) may now be written as

$$0 \geq E_{i+1,j} - E_{i+1,j+1} + \sum_{k=i+1}^{j+1} a_{i,k}(E_{k,j+1}E_{i+1,j} - E_{k,j}E_{i+1,j+1})$$

by substituting $E_{i,j}$ and $E_{i,j+1}$ with their values ((2.7) and (2.8)).

Note that the term corresponding to $k = i+1$ vanishes. Moreover, for $k \geq i+2$ we know that $E_{k,j+1}E_{i+1,j} - E_{k,j}E_{i+1,j+1} \geq 0$ and $a_{i,k} \leq a_{i+1,k}$ (by the induction assumption and Proposition 2.1 respectively).

Therefore, we derive

$$\begin{aligned}
&E_{i+1,j} - E_{i+1,j+1} + \sum_{k=i+1}^{j+1} a_{i,k}(E_{k,j+1}E_{i+1,j} - E_{k,j}E_{i+1,j+1}) \\
&\quad \leq E_{i+1,j} - E_{i+1,j+1} + \sum_{k=i+2}^{j+1} a_{i+1,k}(E_{k,j+1}E_{i+1,j} - E_{k,j}E_{i+1,j+1}) \\
&\quad = E_{i+1,j+1}(\mathbf{1}_{\{j\geq i+2\}} - 1) \\
&\quad \leq 0,
\end{aligned}$$

where the equality follows from straightforward calculations and reduction. Then (2.6) is true, which completes the proof.

Remark 2.2. We have just shown that in the case of an upper triangular generator, if $\boldsymbol{P}_t\boldsymbol{U} \in \mathrm{TP}_2$ for any $0 \leq t$, then $\boldsymbol{G}\boldsymbol{U} = \int_0^{+\infty} \boldsymbol{P}_t\boldsymbol{U}\,\mathrm{d}t \in \mathrm{TP}_2$. We do not know whether this result is still valid without any assumption on the generator.

3. An application to reliability

Let us now apply the previous results to reliability. First, we describe our system.

3.1. Description of the system

We consider a repairable system with a finite state space. Let 1, 2, ..., m be the up-states and $m+1$, ..., $m+p$ be the down-states. The system starts from an up-state. It evolves in time according to a Markov process up to its first failure and it almost surely breaks down after a finite time: $\mathrm{P}_i(T < +\infty) = 1$ for every $i \in \{1, \ldots, m\}$, where T is the first on period of the system. The system evolves according to the same Markov process after any repair. The repair of the system begins as soon as it breaks down and has a random duration that depends neither on the previous evolution of the system nor on the completeness degree of the repair. If the system is in the down-state $m+k$ ($1 \leq k \leq p$), the repair has the same (general) distribution as a random variable R_{m+k}, with a finite mean r_{m+k}. Let $\boldsymbol{r}$ be the $p \times 1$ column matrix of the means r_{m+k}. After any repair, the system starts again from an up-state that is assumed to be independent of the previous evolution of the system (and, consequently, of the down-state by the time of the repair). Then, let $d(i)$ be the probability for the system to start again from state i ($1 \leq i \leq m$) after *any* repair and let $\boldsymbol{d} = (d(1), d(2), \ldots, d(m))$ be the so-called 'start-again' distribution (after repair). Note that the assumption according to which this distribution is the same after any down-state implies that there are some up-states that may be reached by repair from any down-state (they are numbered from 1 up to m_0) and that the support of $\boldsymbol{d}$ is included in $\{1, \ldots, m_0\}$.

Let $(X'_t)_{t\geq 0}$ be the Markov process that describes the evolution of the system up to its first failure:

$$X'_t = \begin{cases} \text{state of the system} & \text{if } t < T, \\ m+k & \text{if } t \geq T \text{ and } X'_T = m+k. \end{cases}$$

(The down-states of the system are absorbing.)

Let $\boldsymbol{A}$ be its generator. The matrix $\boldsymbol{A}$ is subdivided as follows:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \bar{\boldsymbol{0}}^{p,m} & \bar{\boldsymbol{0}}^{p,p} \end{pmatrix}$$

where

$$\boldsymbol{A}_1 = \left(a_{i,j}\right)_{\substack{1\leq i\leq m\\ 1\leq j\leq m}}, \qquad \boldsymbol{A}_2 = \left(a_{i,j}\right)_{\substack{1\leq i\leq m\\ m+1\leq j\leq m+p}}$$

(matrix of the failure rates) and $\bar{\boldsymbol{0}}^{i,j}$ is the $i \times j$ matrix of zeros.

In order to use the results of Section 2, we also consider the process (X''_t) with state space $\{1, \ldots, m+1\}$, which is described in the same way as (X'_t) apart from the fact that the p down-states have been aggregated: the generator $\boldsymbol{A}''$ of (X''_t) is given by $a''_{i,m+1} = \sum_{k=1}^{p} a_{i,m+k}$ and $a''_{i,j} = a_{i,j}$ for $1 \leq i, j \leq m$.

As in Section 2, $\boldsymbol{G}$ is the $m \times m$ matrix such that $\boldsymbol{G}_{i,j} = \int_0^{+\infty} \boldsymbol{P}_t(i,j)\,\mathrm{d}t$, where $\boldsymbol{P}_t(i,j) = \mathrm{P}_i(X'_t = j) = \mathrm{P}_i(X''_t = j)$, for any $1 \leq i, j \leq m$. Let us recall that $\boldsymbol{G} = -\boldsymbol{A}_1^{-1}$.

Finally, for any $n \in \mathbb{N}^*$, let $\bar{\boldsymbol{1}}^n$ be the $n \times 1$ column vector of 1s.

3.2. Computation of the stationary availability

We first compute the stationary availability, that is to say the probability that the system is in an up-state when in long-time run. The point to note is that the process (Z_t) that describes the evolution of the system (with no truncation at time T) is a semi-regenerative process. Indeed, if we have a look at the succession of the new starts after repair (at $T_n,\ n \in \mathbb{N}$), it is readily seen that the later evolution of the system after a new start only depends on the state in which it starts again.

Proposition 3.1. *The stationary availability of the system exists and is*

$$D_\infty(\boldsymbol{d}) = \frac{1}{1 + \boldsymbol{d}_\infty(\boldsymbol{d})}$$

with

$$\boldsymbol{d}_\infty(\boldsymbol{d}) = \frac{\boldsymbol{d}\boldsymbol{G}\boldsymbol{A}_2\boldsymbol{r}}{\boldsymbol{d}\boldsymbol{G}\bar{\boldsymbol{1}}^m}. \tag{3.1}$$

Proof. Let $(Z_{T_n})_{n\in\mathbb{N}}$ be the Markov chain formed by the succession of the states in which the system starts again after repair. With our assumptions, it is clear that $\mathrm{P}(Z_{T_n} = i) = d(i)$ for any $1 \le n,\ 1 \le i \le m$, and the stationary distribution of this Markov chain is $\boldsymbol{d}$.

Also, by reducing the state space to $C = \{i \in \{1, \dots, m\}$ such that $d(i) > 0\}$ if necessary, we may assume that this Markov chain is irreducible. Moreover, the lengths of the cycles of (Z_t) are clearly non-lattice.

Then, general theorems of the Markov renewal theory (see Cocozza-Thivent (1997) or Çinlar (1975), for instance) imply that, if $\sum_{i=1}^m d(i)\mathrm{E}_i(T_1) < +\infty$, the stationary availability of the system exists and is

$$D_\infty(\boldsymbol{d}) = \frac{\sum_{i=1}^m d(i)\mathrm{E}_i(\int_0^{T_1} \mathbf{1}_{\{X_s\in\{1,\dots,m\}\}}\,\mathrm{d}s)}{\sum_{i=1}^m d(i)\mathrm{E}_i(T_1)},$$

where E_i is the conditional expectation given that $Z_0 = i$ and T_1 the duration of the first cycle $(T_0 = 0)$.

Let us recall that T is the first on period of the system and let T_{Rep} be the duration of the repair at the end of the first cycle. With this notation, we have

$$D_\infty(\boldsymbol{d}) = \frac{\sum_{i=1}^m d(i)\mathrm{E}_i(T)}{\sum_{i=1}^m d(i)(\mathrm{E}_i(T) + \mathrm{E}_i(T_{\mathrm{Rep}}))} = \frac{1}{1 + \boldsymbol{d}_\infty(\boldsymbol{d})},$$

with

$$\boldsymbol{d}_\infty(\boldsymbol{d}) = \frac{\sum_{i=1}^m d(i)\mathrm{E}_i(T_{\mathrm{Rep}})}{\sum_{i=1}^m d(i)\mathrm{E}_i(T)}.$$

Moreover, for $1 \le i \le m$, we also have

$$\mathrm{E}_i(T) = \int_0^{+\infty} \mathrm{P}_i(T > t)\,\mathrm{d}t = \int_0^{+\infty} \sum_{j=1}^m \boldsymbol{P}_t(i, j)\,\mathrm{d}t = \sum_{j=1}^m \boldsymbol{G}_{i,j} = (\boldsymbol{G}\bar{\boldsymbol{1}}^m)(i)$$

and

$$\mathrm{E}_i(T_{\mathrm{Rep}}) = \sum_{k=1}^p \mathrm{P}_i(X_{T_1^-} = m + k)\boldsymbol{r}_{m+k}$$

with

$$\mathrm{P}_i(X_{T_1^-} = m + k) = \sum_{j=1}^{m} \int_0^{+\infty} \boldsymbol{P}_t(i, j) a_{j,m+k} \, \mathrm{d}t$$
$$= \sum_{j=1}^{m} \boldsymbol{G}_{i,j} a_{j,m+k} = (\boldsymbol{G}\boldsymbol{A}_2)(i, m + k).$$

We derive the existence of the stationary availability and (3.1) results from straightforward calculations.

Remark 3.1. Note that, from this proof, (3.1) may simply be understood as the usual quotient of the mean down-time by the mean up-time of the system on a cycle.

3.3. Some conditions under which a complete repair is optimal

We now come to our initial problem, as stated in the introduction: we give here conditions under which a complete repair is optimal or, more generally, under which the stationary availability is higher as the repair is complete.

With that aim, we first order the up-states according to their increasing degradation degree, or, more precisely, in such a way that the mean duration of the repair following a breakdown in state i increases with i (for $1 \leq i \leq m$). This is expressed by assuming that the vector $\boldsymbol{A}_2\boldsymbol{r}$ is increasing componentwise. Under this assumption, a repair associated with the 'start-again' distribution $\boldsymbol{d}$ is considered to be more complete than a repair associated with $\boldsymbol{d}_2$ if $\boldsymbol{d}_1$ is smaller than $\boldsymbol{d}_2$, with respect to the *reversed hazard rate ordering*.

Also, the ageing property of our system is translated by assuming that the Markov process (X_t'') that describes the evolution of the system up to its first failure is in $\mathcal{M}_{\mathrm{rh}}^{\mathrm{U}}$. Note that, according to Proposition 2.1, this implies that the 'global' failure rate associated with state i $(\sum_{j=1}^{p} a_{i,m+j})$ is increasing with i for $1 \leq i \leq m$ (or equivalently $\boldsymbol{A}_2\bar{\boldsymbol{1}}^p$ is increasing componentwise). This assumption is quite natural, for the up-states have been ordered according to their increasing degradation degree.

Proposition 3.2. *Let us assume that:*

(H1) *the vector $\boldsymbol{A}_2\boldsymbol{r}$ is increasing componentwise;*

(H2) $(X_t'') \in \mathcal{M}_{\mathrm{rh}}^{\mathrm{U}}$ *(which is equivalent to saying that $\boldsymbol{A}_2\bar{\boldsymbol{1}}^p$ is increasing componentwise and $\boldsymbol{A}_1$ is upper triangular such that $a_{i,j} \leq a_{i+1,j}$ for any $3 \leq i + 2 \leq j \leq m$).*

Then, for any probability row vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ on $\{1, \ldots, m\}$ with support in $\{1, \ldots, m_0\}$

$$\boldsymbol{d}_1 \prec_{\mathrm{rh}} \boldsymbol{d}_2 \Longrightarrow D_\infty(\boldsymbol{d}_1) \geq D_\infty(\boldsymbol{d}_2).$$

In particular, the stationary availability is optimal for a complete repair:

$$D_\infty(\boldsymbol{d}) \leq D_\infty(\boldsymbol{\delta}_1)$$

for any probability row vector $\boldsymbol{d}$ on $\{1, \ldots, m\}$ with support in $\{1, \ldots, m_0\}$, where $\boldsymbol{\delta}_1$ is the Dirac distribution at the perfect working state, denoted by 1.

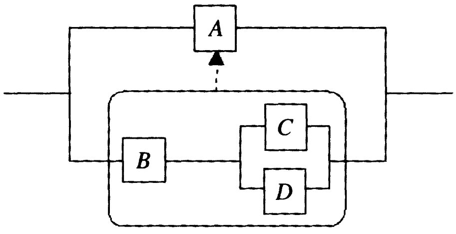

FIGURE 1: Structure of the system.

Proof. Let us take $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ such that $\boldsymbol{d}_1 \prec_{\mathrm{rh}} \boldsymbol{d}_2$ or, equivalently, such that

$$\begin{pmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \end{pmatrix} \boldsymbol{U} \in \mathrm{TP}_2.$$

Note that, as $(X''_t) \in \mathcal{M}^{\mathrm{U}}_{\mathrm{rh}}$, we know from Proposition 2.4 that $\boldsymbol{GU} \in \mathrm{TP}_2$ and $\boldsymbol{U}^{-1}\boldsymbol{GU} \geq 0$. Then, Lemma 2.1 with

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \end{pmatrix}$$

and $\boldsymbol{B} = \boldsymbol{G}$ implies that

$$\begin{pmatrix} \boldsymbol{d}_1\boldsymbol{G} \\ \boldsymbol{d}_2\boldsymbol{G} \end{pmatrix} \boldsymbol{U} \in \mathrm{TP}_2.$$

Now, we may derive from Lemma 2.2 with $\boldsymbol{\xi}_1 = \boldsymbol{d}_1\boldsymbol{G}$, $\boldsymbol{\xi}_2 = \boldsymbol{d}_2\boldsymbol{G}$, $\boldsymbol{z} = \boldsymbol{A}_2\boldsymbol{r}$ and $\boldsymbol{w} = \bar{\boldsymbol{1}}^m$ (with the help of (H1)) that

$$\frac{\boldsymbol{d}_1\boldsymbol{G}\boldsymbol{A}_2\boldsymbol{r}}{\boldsymbol{d}_1\boldsymbol{G}\bar{\boldsymbol{1}}^m} \leq \frac{\boldsymbol{d}_2\boldsymbol{G}\boldsymbol{A}_2\boldsymbol{r}}{\boldsymbol{d}_2\boldsymbol{G}\bar{\boldsymbol{1}}^m},$$

which means that $d_\infty(\boldsymbol{d}_1) \leq d_\infty(\boldsymbol{d}_2)$ (see (3.1)).

The second point is straightforward, since $\boldsymbol{\delta}_1 \prec_{\mathrm{rh}} \boldsymbol{d}$ for any $\boldsymbol{d}$, which completes the proof.

We now end our study with two examples that show that the usual stochastic ordering is adapted neither to modelling the ageing property of our system nor to measuring the completeness degree of the repair in order to get the desired property, that is the more complete the repair, the higher the stationary availability.

In both examples, we consider a system composed of four components A, B, C and D, with respective constant failure rates λ_A, λ_B, λ_C and λ_D. Each component may be repaired when the system is down, but none when the system is up.

Example 3.1. Here, component A and the sub-system composed of B, C and D (see Figure 1) are in stand-by redundancy: at first, component A is active and the sub-system is waiting. When component A fails, the sub-system is activated. Component C starts with probability γ_C. Components B and D always start.

The up-states are: $1 = A(BCD)_{\mathrm{w}}$, $2 = \bar{A}BCD$, $3 = \bar{A}BC\bar{D}$ and $4 = \bar{A}B\bar{C}D$, where the symbol A means that component A is active, and $\bar{A}$ means that it has failed. We use the same notation for components B, C and D. The notation $(BCD)_{\mathrm{w}}$ means that the sub-system composed of B, C and D is inactive (or 'waiting').

The down-states now are: $5 = \bar{A}\bar{B}CD, 6 = \bar{A}\bar{B}C\bar{D}, 7 = \bar{A}B\bar{C}\bar{D}, 8 = \bar{A}\bar{B}\bar{C}D$.

Here, each of the four down-states may lead to the up-states 1 and 2 by repair and $m_0 = 2$. We have

$$A_1 = \begin{pmatrix} -\lambda_A & \lambda_A\gamma_C & 0 & \lambda_A(1-\gamma_C) \\ 0 & -(\lambda_B+\lambda_C+\lambda_D) & \lambda_D & \lambda_C \\ 0 & 0 & -(\lambda_B+\lambda_C) & 0 \\ 0 & 0 & 0 & -(\lambda_B+\lambda_D) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda_B & 0 & 0 & 0 \\ 0 & \lambda_B & \lambda_C & 0 \\ 0 & 0 & \lambda_D & \lambda_B \end{pmatrix}.$$

We take $\lambda_C \le \lambda_B$, so that $A_2\bar{\mathbf{1}}^p$ is *increasing*, and $\lambda_A(1-\gamma_C) \le \lambda_C + \lambda_B$ so that the *aggregated process* (X_t'') is *monotone with respect to the usual stochastic ordering* (with an upper triangular generator). (See Kijima (1997) for a characterization of a monotone Markov process with respect to the usual stochastic ordering in terms of its generator.)

There is a single repairman and the duration for the repair of A is negligible in front of the others. Then, the mean duration of the repair is independent of the state (1 or 2) in which the system starts again.

Numerically, we take $\lambda_A = 1, \lambda_B = 0.8, \lambda_C = 0.1, \lambda_D = 0.1$ and $\gamma_C = 0.1$.

The mean durations for the repair of B, C and D respectively are 0.001, 0.1 and 0.01. We derive:

$$\boldsymbol{r} = \begin{pmatrix} 0.001 \\ 0.011 \\ 0.11 \\ 0.101 \end{pmatrix} \quad \text{and} \quad A_2\boldsymbol{r} = \begin{pmatrix} 0 \\ 0.0008 \\ 0.0198 \\ 0.0918 \end{pmatrix}$$

and (H1) is true. After computation, we also get

$$D_\infty(\boldsymbol{\delta}_1) = 0.9580 < D_\infty(\boldsymbol{\delta}_2) = 0.9893$$

(and so it is better not to repair component A).

As $\boldsymbol{\delta}_1$ is smaller than $\boldsymbol{\delta}_2$ for most stochastic orderings (actually, we do not know of any counterexample), we find that, under the assumptions of this example, $\boldsymbol{d}_1$ smaller than $\boldsymbol{d}_2$ does not imply that $D_\infty(\boldsymbol{d}_1) \ge D_\infty(\boldsymbol{d}_2)$, whatever the stochastic ordering notion used to compare $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ may be. Then, the Markov process needs to be monotone with respect to a stronger stochastic ordering than the usual one and (H2) seems to be required.

Example 3.2. Here, component A, component B and the sub-system composed of components C and D in series (see Figure 2) are in standby-redundancy. At first, component A is active. When component A fails, component B is activated and starts with a probability γ_B. When component B fails (or when it refuses to start), the sub-system composed of components C and D is activated. Component C starts with probability γ_C and component D with probability γ_D.

The up-states are $1 = AB_{\mathrm{w}}(CD)_{\mathrm{w}}, 2 = \bar{A}B(CD)_{\mathrm{w}}, 3 = \bar{A}\bar{B}CD$, and the down-states are $4 = \bar{A}\bar{B}\bar{C}D, 5 = \bar{A}\bar{B}C\bar{D}, 6 = \bar{A}\bar{B}\bar{C}\bar{D}$. Each of the three down-states leads to the three

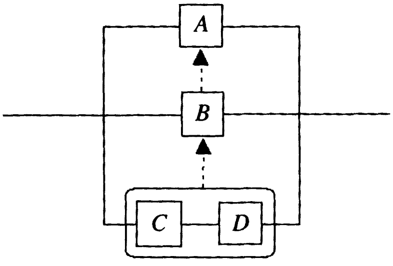

FIGURE 2: Structure of the system.

up-states by repair ($m_0 = m = 3$). We have

$$A_1 = \begin{pmatrix} -\lambda_A & \lambda_A\gamma_B & \lambda_A(1-\gamma_B)\gamma_C\gamma_D \\ 0 & -\lambda_B & \lambda_B\gamma_C\gamma_D \\ 0 & 0 & -(\lambda_C+\lambda_D) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} a_{14} & a_{15} & a_{16} \\ \lambda_B(1-\gamma_C)\gamma_D & \lambda_B\gamma_C(1-\gamma_D) & \lambda_B(1-\gamma_C)(1-\gamma_D) \\ \lambda_C & \lambda_D & 0 \end{pmatrix},$$

where

$$a_{14} = \lambda_A(1-\gamma_B)(1-\gamma_C)\gamma_D,$$
$$a_{15} = \lambda_A(1-\gamma_B)\gamma_C(1-\gamma_D),$$
$$a_{16} = \lambda_A(1-\gamma_B)(1-\gamma_C)(1-\gamma_D).$$

We take: $\lambda_A = 4$, $\lambda_B = 5.6$, $\lambda_C = 3.9$, $\lambda_D = 3.9$, $\gamma_B = 0.3$, $\gamma_C = 0.2$ and $\gamma_D = 0.4$. We then get

$$A_1 = \begin{pmatrix} -4 & 1.2 & 0.224 \\ 0 & -5.6 & 0.448 \\ 0 & 0 & -7.80 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.896 & 0.336 & 1.344 \\ 1.792 & 0.672 & 2.688 \\ 3.900 & 3.900 & 0 \end{pmatrix}$$

and

$$A_2\bar{\mathbf{1}}^p = \begin{pmatrix} 2.576 \\ 5.152 \\ 7.800 \end{pmatrix}.$$

There is a single repairman and the durations for the repairs of A and B are negligible in front of the others. The mean durations for the repairs of C and D are, respectively, 0.01 and 0.015. We compute

$$\boldsymbol{r} = \begin{pmatrix} 0.01 \\ 0.015 \\ 0.025 \end{pmatrix} \quad \text{and} \quad A_2\boldsymbol{r} = \begin{pmatrix} 0.0476 \\ 0.0952 \\ 0.0975 \end{pmatrix}.$$

Then, (H1) and (H2) are true (and the results of Proposition 3.2 are valid).

We take $\boldsymbol{d}_1 = [0.5, 0.5, 0]$ and $\boldsymbol{d}_2 = [0.5, 0, 0.5]$. It is easy to check that $\boldsymbol{d}_1 \prec_{\mathrm{st}} \boldsymbol{d}_2$, but that $\boldsymbol{d}_1 \not\prec_{\mathrm{rh}} \boldsymbol{d}_2$ and we get $D_\infty(\boldsymbol{d}_1) = 0.9332 < D_\infty(\boldsymbol{d}_2) = 0.9355$.

We now take $\boldsymbol{d}_1 = [\frac{1}{3}, \frac{2}{5}, \frac{4}{15}]$ and $\boldsymbol{d}_2 = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ (Kijima (1997, Example 3.8)). We now have $\boldsymbol{d}_1 \prec_{\mathrm{hr}} \boldsymbol{d}_2$, but $\boldsymbol{d}_1 \not\prec_{\mathrm{rh}} \boldsymbol{d}_2$, where $\prec_{\mathrm{hr}}$ represents ordering with respect to the hazard rate. Here we get $D_\infty(\boldsymbol{d}_1) = 0.9284 < D_\infty(\boldsymbol{d}_2) = 0.9286$.

We derive from this example that, even assuming (H1) and (H2), neither $\boldsymbol{d}_1 \prec_{\mathrm{st}} \boldsymbol{d}_2$ nor $\boldsymbol{d}_1 \prec_{\mathrm{hr}} \boldsymbol{d}_2$ is sufficient to deduce that $D_\infty(\boldsymbol{d}_1) \geq D_\infty(\boldsymbol{d}_2)$, which confirms the accuracy of our assumption that $\boldsymbol{d}_1 \prec_{\mathrm{rh}} \boldsymbol{d}_2$.

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