Stationary Availability of a Markov System with Preventive Random Maintenance

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Abstract : We consider a system that can be up or down at time t. This system is assumed to be reparable, with a finite state space, and to evolve among the up states according to a Markov process. A repair begins as soon as the system is down and has a random duration with a general distribution. This system is subjected to a preventive maintenance policy : the system is instantaneously inspected at random times until it is found in such a "bad" state that we stop it to maintain it, or until it is found down, being repaired, whichever occurs first. The random inspection times depend on the successive states in which the system is found when inspected. The random duration of a maintenance action depends on the degradation state of the system. We compute the stationary availability of the maintained system and we give a sufficient condition for the preventive maintenance policy to improve the stationary availability. We show in a particular case that the optimisation of the preventive maintenance policy may be restricted to the maintenance policies with deterministic inter-inspections intervals. We observe the same property on other examples.

1 Description of the System - Notations

We consider a reparable system that evolves among the up-states according to a Markov process. Let "1", "2", ..., "m" be the up-states. (One can imagine, for example, that the states "1" to "m" correspond to some increasing degradation of the system). We assume that the system has a single down state, denoted by "$m+1$". At the beginning, the system is up. Let T be the first on-period of the system and let $\left(X_t^1\right)_{t\geq 0}$ be the process that describes the evolution of the system up to the first failure :

$$X_t^1 = \begin{cases} \text{state of the system} & \text{if } t \leq T, \\ m+1 & \text{if } t > T. \end{cases}$$

$\left(X_t^1\right)$ is a *Markov process.* We assume that the system almost surely breaks down after a finite time : $\mathbb{P}_i\left(T < +\infty\right) = 1$ for every $i \in \{1, ..., m\}$. When the system goes down, a repair is begun. If the system is in state $i \in \{1, ..., m\}$ by the time of the breakdown, the repair has a random duration that does not depend on the previous evolution of the system except from state i and has the same (general) distribution as a random variable R_i, with a finite mean. After a repair, the system is in an up state that does not depend on the previous evolution of the system. For $i \in \{1, ..., m\}$, let $D_R\left(i\right)$ be the probability that the system is in state i after a repair and let $D_R = \left(D_R\left(1\right), ..., D_R\left(m\right)\right)$.

This system is subjected to the following *preventive maintenance policy.*

Let U_1, U_2, ..., U_m be non-negative random variables with ρ_1, ρ_2, ..., ρ_m as respective distributions and finite positive expectations. Let p be a fixed integer, $1 \leq p \leq m-1$, and let M_{p+1}, M_{p+2}, ..., M_m be random variables with finite expectations.

The system is *instantaneously* inspected at times S_1, S_2, ..., S_n, ... recursively defined by : S_1 is a random variable independent on the evolution of the system (except from state X_0^1), with $\rho_{X_0^1}$ for distribution and, for $n \in \mathbb{N}^*$,

- If $X_{S_n}^1 \in \{1, ..., p\}$, the system is in a "good" state. We leave it evoluate alone and it is inspected once again at time $S_{n+1} = S_n + U^{(n)}$, where $U^{(n)}$ is a random variable independent on the evolution of the system before S_n (except from state $X_{S_n}^1$), with $\rho_{X_{S_n}^1}$ for distribution.

- If $X_{S_n}^1 \in \{p+1, ..., m\}$, the system is in a "bad" state. It is stopped to be maintained. The maintenance action lasts for a random duration that is independent on the evolution of the system before S_n (except from state $X_{S_n}^1$), with the same distribution as $M_{X_{S_n}^1}$.

- If $X_{S_n}^1 = m+1$, the system is down, being repaired. The repair is carried out up to its end with no further inspection.

In the same way as after a repair, we assume that after a maintenance action, the system is in an up state that does not depend on the previous evolution of the system. For $i \in \{1, ..., m\}$, let $D_M(i)$ be the probability that the system is in state i after a maintenance action and let $D_M = (D_M(1), ..., D_M(m))$.

After a repair or a maintenance action (a *down* period), the *sequence of the inspections is renewed*, which means that we start again with a new sequence of inspections, recursively defined as above.

Let "$m+2$" be the maintenance state and $(X_t)_{t\geq 0}$ be the process that describes the evolution of the maintained system, which takes its values in $\{1, ..., m+2\}$. We can notice that, after a down period, the later evolution of the maintained system only depends on the state in which the system starts again, so that (X_t) is a *semi-regenerative process*. This is the basic remark to compute the stationary availability.

Matrix Notations :

- For any $k, n \in \mathbb{N}^*$, I_n is the $n \times n$ identity matrix, $\bar{0}^{k,n}$ is the $k \times n$ matrix of zeros.
- b and μ are the $m \times m$ matrices such that $b_{i,j} = \mathbb{P}_i\left(X^1_{U_i} = j\right)$ and $\mu_{i,j} = \mathbb{P}_i\left(X^1_{T^-} = j\right)$ for any $i, j \in \{1, ..., m\}$. The matrix b is subdivised as follows :

$$b = \begin{pmatrix} \bar{b}^{p,p} & \bar{b}^{p,m-p} \\ \bar{b}^{m-p,p} & \bar{b}^{m-p,m-p} \end{pmatrix}.$$

- One may check that 1 is not an eigenvalue of $\bar{b}^{p,p}$ so that we may introduce the matrix B such that

$$B = \begin{pmatrix} \left(I_p - \bar{b}^{p,p}\right)^{-1} & \bar{0}^{p,m-p} \\ \bar{b}^{m-p,p}\left(I_p - \bar{b}^{p,p}\right)^{-1} & I_{m-p} \end{pmatrix}.$$

- $$\overline{\mathbb{E}(M_\bullet)} = \begin{pmatrix} \bar{0}^{p,1} \\ \mathbb{E}(M_{p+1}) \\ \mathbb{E}(M_{p+2}) \\ . \\ \mathbb{E}(M_m) \end{pmatrix}, \overline{\mathbb{E}(R_\bullet)} = \begin{pmatrix} \mathbb{E}(R_1) \\ \mathbb{E}(R_2) \\ . \\ \mathbb{E}(R_m) \end{pmatrix}, \overline{\mathbb{E}_\bullet(T)} = \begin{pmatrix} \mathbb{E}_1(T) \\ \mathbb{E}_2(T) \\ . \\ \mathbb{E}_m(T) \end{pmatrix}, \overline{\mathbb{E}_\bullet(R)} = \begin{pmatrix} \mathbb{E}_1(R) \\ \mathbb{E}_2(R) \\ . \\ \mathbb{E}_m(R) \end{pmatrix},$$

  where $\mathbb{E}_i$ is the expectation with respect to the conditional distribution $\mathbb{P}_i(.) = \mathbb{P}_i\left(./X^1_0 = i\right)$. If we look after the cycles of the unmaintained system that begin with the re-startings of the system after a repair, $\mathbb{E}_i(T)$ is the mean duration of the up-period which takes place at the beginning of a cycle that starts in state i, when $\mathbb{E}_i(R)$ is the mean duration of the repair which takes place at the end of the same cycle. (One may check that $\overline{\mathbb{E}_\bullet(R)} = \mu\overline{\mathbb{E}(R_\bullet)}$).

2 Computation of the Stationary Availability

Let D_∞ be the stationary availability of the maintained system. Let us recall that :

$$D_\infty = \lim_{t\to+\infty} \sum_{k=1}^{m} \mathbb{P}(X_t = k), \text{ whenever it exists.}$$

Theorem 1 *The stationary availability of the maintained system exists and is*

$$D_\infty = \frac{1}{1+d_\infty} \text{ with } d_\infty = \frac{D_{MR}B\left((I_m - b)\,\mu\overline{\mathbb{E}(R_\bullet)} + b\overline{\mathbb{E}(M_\bullet)}\right)}{D_{MR}B\,(I_m - b)\,\overline{\mathbb{E}_\bullet(T)}} \tag{1}$$

and

$$D_{MR} = \left[D_M B\,(I_m - b)\,\bar{1}^m\right] D_R + \left[1 - D_R B\,(I_m - b)\,\bar{1}^m\right] D_M.$$

Remark 1 *All the terms of d_∞ may easily be computed from the data. Indeed, if A^1 is the generative matrix of the Markov process $\left(X^1_t\right)$ and A is the matrix A^1 truncated at order m, one may check that $\mu = -A^{-1}diag(A^1(1, m+1),\ A^1(2, m+1),\ ...,\ A^1(m, m+1))$, and that $\overline{\mathbb{E}_\bullet(T)} = -A^{-1}\bar{1}^m$. Moreover, $b_{i,j} = \int_0^{+\infty} P_t(i,j)\,\rho_i(dt)$ and the computation of the $P_t(i,j)$ have already been much studied in the literature (cf [1] e.g.).*

3 A Sufficient Condition for the Preventive Maintenance Policy to Improve the Stationary Availability

Let D_∞^{ini} be the stationary availability of the initial (i.e. unmaintained) system ($D_\infty^{ini} = \frac{1}{1+d_\infty^{ini}}$ with $d_\infty^{ini} = \frac{D_R.\overline{\mathbb{E}_\bullet(R)}}{D_R.\overline{\mathbb{E}_\bullet(T)}}$). For $k \in \{p+1, ..., m\}$, let D_k be the k^{th} row of I_m.

For any probability vector D on $\{1, ..., m\}$, let $D_\infty^{ini}(D)$ be the stationary availability of the initial system if new starts of the initial system after a repair are controlled by D instead of D_R.

We show that, if new starts after a maintenance action (controlled by D_M) are "at least as good" as after a repair $\left(D_\infty^{ini}(D_M) \geq D_\infty^{ini}\right)$, and if new starts after a repair are "better" than new starts in state k for any $k \in \{p+1, ..., m\}$, $\left(D_\infty^{ini} \geq D_\infty^{ini}(D_k)\right)$, then, if the maintenance actions are not too long in average, the maintenance policy improves the stationary availability.

Theorem 2 *Let us assume that*

- $D_\infty^{ini}(D_M) \geq D_\infty^{ini}$,
- $D_\infty^{ini} \geq D_\infty^{ini}(D_k)$, *for any* $k \in \{p+1, ..., m\}$.

Then, if $\mathbb{E}(M_k) \leq \mathbb{E}_k(R) - d_\infty^{ini}.\mathbb{E}_k(T)$, *for any* $k \in \{p+1, ..., m\}$, *we have* $D_\infty \geq D_\infty^{ini}$.

4 Optimisation of the Preventive Maintenance Policy Under Specific Assumptions

Our problem is here to see whether there is a preventive maintenance policy that makes the stationary availability maximal. As far as the durations of the maintenance actions are concerned, it is easy to see on (1) that the stationary availability depends on them only through their means and that, as expected, *the stationary availability is decreasing with each* $\mathbb{E}(M_i)$, *for any* $i \in \{p+1, ..., m\}$. The real problem is to study the influence of the distributions of the inter-inspection intervals ρ_1, ρ_2, ..., ρ_m on the stationary availability. We deal here with this problem under specific assumptions under which we show that *the optimization study may be restricted to the preventive maintenance policies with deterministic inter-inspection intervals.*

To indicate the dependence on the distributions ρ_1, ρ_2, ..., ρ_m, D_∞ and d_∞ are now respectively denoted by $D_\infty(\rho_1, \rho_2, ..., \rho_m)$ and $d_\infty(\rho_1, \rho_2, ..., \rho_m)$.

We assume here that *the system starts again in the same way after a maintenance action as after a repair* ($D_M = D_R$). Moreover, *if the states 1 to p correspond to some increasing degradation of the system, we assume that the system may only go worse as long as it is in* $\{1, ..., p\}$ (the generative matrix of the Markov process $\left(X_t^1\right)$ truncated at order p, say $\bar{A}^{p,p}$, is upper triangular).

Theorem 3 *Under these assumptions ($D_M = D_R$ and $\bar{A}^{p,p}$ upper triangular) :*

1°) $\left(\textit{There exist some distributions } \rho_1^0, \rho_2^0, ..., \rho_m^0 \textit{ such that } D_\infty\left(\rho_1^0, \rho_2^0, ..., \rho_m^0\right) > D_\infty^{ini}\right) \quad (H)$

$$\Updownarrow$$

$\left(\textit{There exist } c_1^0, c_2^0, ..., c_m^0 > 0 \textit{ such that } D_\infty\left(\delta_{c_1}^0, \delta_{c_2}^0, ..., \delta_{c_m}^0\right) > D_\infty^{ini}\right) \quad (H')$

2°) Under the assumption (H) *or* (H'), *there exist* c_1^{opt}, c_2^{opt}, ..., c_m^{opt} *such that*

$$D_\infty\left(\delta_{c_1^{opt}}, \delta_{c_2^{opt}}, ..., \delta_{c_m^{opt}}\right) \geq D_\infty(\rho_1, \rho_2, ..., \rho_m)\,, \textit{ for any distributions } \rho_1, \rho_2, ..., \rho_m.$$

5 An Example

The initial system is a "k out of n system". It is composed with n identical components with constant failure rate λ and repair rate μ. The system is up if and only if k components are working. For $i \in \{1, ..., n\}$, let i be the state where exactly $i-1$ components are down. There are $m = n-k+1$ up-states. The repairs and the maintenance actions are assumed to put the system back to the new state (i.e. state "1"), so that we have $D_M = D_R = (1, 0..., 0)$.

Note that the assumption "$\bar{A}^{p,p}$ upper triangular" of **theorem 3** is *not* true.

We take $k=2$, $\lambda=1$, $\mu=2$, $\mathbb{E}(R_m)=\frac{m}{10}$ (the system may only go down from state m) and $\mathbb{E}(M_j)=\frac{j}{100}$, for any $j\in\{p+1,...,m\}$.

We first check the advisability of our preventive maintenance policy with the help of **theorem 2**. As the other assumptions are true, we only have to compare $\mathbb{E}(M_k)$ to $x_k=\mathbb{E}_k(R)-d_\infty^{ini}.\mathbb{E}_k(T)$ for any $k\in\{2,...n-1\}$ (cf **table 1**). Then, we compute the optimal distributions for the inter-inspection intervals. As we cannot consider all the possible distributions, we assume that they are GAMMA distributions. We use the optimisation tools of MATLAB to find the best parameters of those GAMMA distributions and we find that those parameters correspond to very small standard deviations so that the *best inter-inspection intervals are deterministic*. **Table 2** gives the stationary availability of the initial system $\left(D_\infty^{ini}\right)$ and the optimal stationary availability of the maintained system (D_∞) for $p\in\{1,...,n-2\}$.

n		k=2	k=3	k=4	k=5	k=6	k=7	k=8
3	x_k	0.0571						
	$E(M_k)$	0.02						
4	x_k	0.0333	0.1000					
	$E(M_k)$	0.02	0.03					
5	x_k	0.0188	0.0518	0.1271				
	$E(M_k)$	0.02	0.03	0.04				
6	x_k	0.0101	0.0262	0.0573	0.1398			
	$E(M_k)$	0.02	0.03	0.04	0.05			
7	x_k	0.0051	0.0126	0.0258	0.0544	0.1424		
	$E(M_k)$	0.02	0.03	0.04	0.05	0.06		
8	x_k	0.0024	0.0058	0.0113	0.0217	0.0474	0.1392	
	$E(M_k)$	0.02	0.03	0.04	0.05	0.06	0.07	
9	x_k	0.0011	0.0026	0.0048	0.0087	0.0168	0.0396	0.1341
	$E(M_k)$	0.02	0.03	0.04	0.05	0.06	0.07	0.08

table 1

n	D_∞^{ini}	p=1	p=2	p=3	p=4	p=5	p=6	p=7
3	0.8537	0.9434						
4	0.8824	0.9326	0.9615					
5	0.9140	0.9372	0.9515	0.9712				
6	0.9431	0.9470	0.9519	0.9627	0.9789			
7	0.9658	D_∞^{ini}	D_∞^{ini}	D_∞^{ini}	0.9703	0.9853		
8	0.9812	D_∞^{ini}	D_∞^{ini}	D_∞^{ini}	D_∞^{ini}	D_∞^{ini}	0.9904	
9	0.9903	D_∞^{ini}	D_∞^{ini}	D_∞^{ini}	D_∞^{ini}	D_∞^{ini}	D_∞^{ini}	0.9942

table 2

This example allows us to conclude that, as expected, **theorem 2** provides us with *a sufficient but not necessary condition* for the maintenance policy to improve the stationary availablity (see $n=6$, $p=1$ e.g.). Though, we can see that it gives *a rather good numerical bound for the maximal mean durations of the maintenance actions.*

Though the assumptions of **theorem 3** are not true for this example, the results are still valid and it is sufficient to study the deterministic maintenance policies to optimize the stationary availability. Actually, we have checked the same property on a few other examples, so that we may conjecture that the results of **theorem 3** are always true : *it seems to be always sufficient to restrict the study to the deterministic inter-inspection intervals to optimize the stationary availability.*

References

References

[1] E. Cinlar, *Introduction to stochastic processes,* Prentice-Hall, 1975.

[2] C. Cocozza-Thivent, *Processus stochastiques et fiabilité des systèmes,* Math. et Appl. n°28, Springer, 1997.