Importance Factors in Dynamic Reliability

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ABSTRACT: In dynamic reliability, the evolution of a system is governed by a piecewise deterministic Markov process, which is characterized by different input data. Assuming such data to depend on some parameter $p \in P$, our aim is to compute the first-order derivative with respect to each $p \in P$ of some functionals of the process, which may help to rank input data according to their relative importance, in view of sensitivity analysis. The functionals of interest are expected values of some function of the process, cumulated on some finite time interval [0,t], and their asymptotic values per unit time. Typical quantities of interest hence are cumulated (production) availability, or mean number of failures on some finite time interval and similar asymptotic quantities. The computation of the first-order derivative with respect to $p \in P$ is made through a probabilistic counterpart of the adjoint point method, from the numerical analysis field. Examples are provided, showing the good efficiency of this method, especially in case of large P.

1 INTRODUCTION

In reliability, one of the most common model used in an industrial context for the time evolution of a system is a pure jump Markov process with finite state space. This means that the transition rates between states (typically failure rates, repair rates) are assumed to be constant and independent on the possible evolution of the environment (temperature, pressure, ...). However, the influence of the environment can clearly not always be neglected: for instance, the failure rate of some electronic component may be much higher in case of high temperature. Similarly, the state of the system may influence the evolution of the environmental condition: think for instance of a heater which may be on or off, leading to an increasing or decreasing temperature. Such observations have led to the development of new models taking into account such interactions. In this way, Jacques Devooght introduced in the 90's what he called dynamic reliability, with models issued at the beginning from the domain of nuclear safety, see (Devooght 1997) with references therein. In the probability vocabulary, such models correspond to piecewise deterministic Markov processes (PDMP), introduced by (Davis 1984). Such processes are denoted by $(I_t, X_t)_{t\geq 0}$ in the following. They are hybrid processes, in the sense that both components are not of the same type: the first one I_t is discrete, with values in a finite state space E. Typically, it indicates the state (up/down) for each component of the system at time t, just as for a usual pure jump Markov process. The second component X_t takes its values in a Borel set $V \subset \mathbb{R}^d$ and stands for the environmental conditions (temperature, pressure, ...). The process $(I_t, X_t)_{t\geq 0}$ jumps at countably many random times and both components interact one in each other, as required for models from dynamic reliability: by a jump from $(I_{t^-}, X_{t^-}) = (i, x)$ to $(I_t, X_t) = (j, y)$ (with (i, x), $(j, y) \in E \times V$), the transition rate between the discrete states i and j depends on the environmental condition x just before the jump and is a function $x \longmapsto a(i, j, x)$. Similarly, the environmental condition just after the jump X_t is distributed according to some distribution $\mu_{(i,j,x)}(\mathrm{d}y)$, which depends on both components just before the jump (i, x) and on the after jump discrete state j. Between jumps, the discrete component I_t is constant, whereas the evolution of the environmental condition X_t is deterministic, solution of a set of differential equations which depends on the fixed discrete state: given that $I_t(\omega) = i$ for all $t \in [a, b]$, we have $\frac{d}{dt}X_t(\omega) = \mathbf{v}(i, X_t(\omega))$ for all $t \in [a, b]$, where $\mathbf{v}$ is a mapping from $E \times V$ to V. Contrary to the general model from (Davis 1984), we do not take here into account jumps of $(I_t, X_t)_{t\geq 0}$, eventually entailed by the reaching of the frontier of V.

Given such a PDMP $(I_t, X_t)_{t\geq 0}$, we are interested in different quantities linked to this process, which may be written as cumulated expectations on some time interval $[0,t]$ of some bounded measurable function h of the process:

$$R_{\rho_0}(t) = \mathbb{E}_{\rho_0}\left(\int_0^t h(I_s,X_s)\,\mathrm{d}s\right)$$

where ρ_0 is the initial distribution of the process. Such quantities include e.g. cumulative availability or production availability on some time interval $[0,t]$, mean number of failures on $[0,t]$, mean time spent by $(X_s)_{0\le s\le t}$ on $[0,t]$ between two given bounds...

For such types of quantity, our aim is to study their sensitivity with respect of different parameters $p\in P$, from which may depend both the function h and the input data of the process $(I_t,X_t)_{t\ge 0}$. More specifically, the point is to study the influence of variations of $p\in P$ on $R_{\rho_0}(t)$, through the computation of the first-order derivative of $R_{\rho_0}(t)$ with respect to each $p\in P$. In view of comparing the results for different $p\in P$, we prefer to normalize such derivatives, and we are actually interested in computing the dimensionless first-order logarithmic derivative of $R_{\rho_0}(t)$ with respect to p :

$$IF_p(t) = \frac{p}{R_{\rho_0}(t)}\frac{\partial R_{\rho_0}(t)}{\partial p}$$

which we call importance factor of parameter p in $R_{\rho_0}(t)$. In view of long time analysis, we also want to compute its limit $IF_p(\infty)$, with

$$IF_p(\infty) = \lim_{t\to+\infty}\frac{p}{R_{\rho_0}(t)/t}\frac{\partial\left(R_{\rho_0}(t)/t\right)}{\partial p}$$

Noting that $IF_p(t)$ and $IF_p(\infty)$ only make sense when considering never vanishing parameter p, we consequently assume p to be positive.

This kind of sensitivity analysis was already studied in (Gandini 1990) and in (Cao and Chen 1997) for pure jump Markov processes with countable state space, and extended to PDMP in (Mercier and Roussignol 2007), with more restrictive a model than in the present paper however.

Since the marginal distributions of the process $(I_t,X_t)_{t\ge 0}$ are, in some sense, the weak solution of linear first order hyperbolic equations (Cocozza-Thivent, Eymard, Mercier, and Roussignol 2006), the expressions for the derivatives of the mathematical expectations can be obtained by solving the dual problem (adjoint point method), as suggested in (Lions 1968) for a wide class of partial differential equations. We show here that the resolution of the dual problem provides an efficient numerical method, when the marginal distributions of the PDMP are approximated using a finite volume method.

Due to the reduced size of the present paper, all proofs are omitted and will be provided in a forthcoming paper.

2 ASSUMPTIONS

The jump rates $a(i,j,x)$, the jump distribution $\mu_{(i,j,x)}$, the velocity field $\mathbf{v}(i,x)$ and the function $h(i,x)$ are assumed to depend on some parameter p, where p belongs to an open set $O\subset\mathbb{R}$ or $\mathbb{R}^k$. All the results are written in the case where $O\subset\mathbb{R}$ but extension to the case $O\subset\mathbb{R}^k$ is straightforward. We add exponent $^{(p)}$ to each quantity depending on p, such as $h^{(p)}$ or $R^{(p)}_{\rho_0}(t)$.

We denote by $\rho_t^{(p)}(i,dx)$ the distribution of the process $\left(I_t^{(p)},X_t^{(p)}\right)_{t\ge 0}$ at time t with initial distribution ρ_0 (independent on p). We then have:

$$\begin{aligned}R^{(p)}_{\rho_0}(t) &= \int_0^t \rho_s^{(p)} h^{(p)}\,\mathrm{d}s\\ &= \sum_{i\in E}\int_V\left(\int_0^t h^{(p)}(i,x)\,\mathrm{d}s\right)\rho_s^{(p)}(i,\mathrm{d}x)\end{aligned}$$

In order to prove existence and to calculate derivatives of the functional $R^{(p)}_{\rho_0}$, we shall need the following assumptions ($\mathcal{H}_1$): for each p in O, there is some neighborhood $N(p)$ of p in O such that, for all $i,j\in E\times E$,

- the function $(x,p)\longmapsto a^{(p)}(i,j,x)$ is bounded on $V\times N(p)$, belongs to $C_2(V\times O)$ (twice continuously differentiable on $V\times O$), with all partial derivatives uniformly bounded on $V\times N(p)$,
- for all function $f^{(p)}(x)\in C_2(V\times O)$, with all partial derivatives uniformly bounded on $V\times N(p)$, the function $(x,p)\longmapsto\int f^{(p)}(y)\ \mu^{(p)}_{(i,j,x)}(dy)$ belongs to $C_2(V\times O)$, with all partial derivatives uniformly bounded on $V\times N(p)$,
- the function $(x,p)\longmapsto\mathbf{v}^{(p)}(i,x)$ is bounded on $V\times N(p)$, belongs to $C_2(V\times O)$, with all partial derivatives uniformly bounded on $V\times N(p)$,
- the function $(x,p)\longmapsto h^{(p)}(i,x)$ is bounded on $V\times N(p)$, almost surely (a.s.) twice continuously differentiable on $V\times O$ with a.s. uniformly bounded partial derivatives on $V\times N(p)$, where a.s. means with respect to Lebesgue measure in x.

In all the paper, under assumptions $\mathcal{H}_1$, for each p in O, we shall refer to a $N(p)$ fulfilling the four points of the assumption without any further notice. We recall that under assumptions $\mathcal{H}_1$ (and actually under much milder assumptions), the process $(I_t, X_t)_{t\geq 0}$ is a Markov process, see (Davis 1984) e.g.. Its transition probability distribution is denoted by $P_t^{(p)}(i, x, j, \mathrm{d}y)$.

3 TRANSITORY RESULTS

We first introduce the infinitesimal generators of both Markov processes $(I_t, X_t)_{t\geq 0}$ and $(I_t, X_t, t)_{t\geq 0}$:

Definition 1 *Let $\mathcal{D}_{H_0}$ be the set of functions $f(i,x)$ from $E \times V$ to $\mathbb{R}$ such that for all $i \in E$ the function $x \longmapsto f(i,x)$ is bounded, continuously differentiable on V and such that the function $x \longmapsto \mathbf{v}^{(p)}(i,x) \cdot \nabla f(i,x)$ is bounded on V. For $f \in \mathcal{D}_{H_0}$, we define*

$$H_0^{(p)} f(i,x) = \sum_{j\in E} a^{(p)}(i,j,x) \int f(j,y)\, \mu^{(p)}_{(i,j,x)}(\mathrm{d}y) + \mathbf{v}^{(p)}(i,x) \cdot \nabla f(i,x)$$

where we set $a^{(p)}(i,i,x) = -\sum_{j\neq i} a^{(p)}(i,j,x)$ and $\mu^{(p)}_{(i,i,x)} = \delta_x$.

Let $\mathcal{D}_H$ be the set of functions $f(i,x,s)$ from $E \times V \times \mathbb{R}_+$ to $\mathbb{R}$ such that for all $i \in E$ the function $(x,s) \longmapsto f(i,x,s)$ is bounded, continuously differentiable on $V \times \mathbb{R}_+$ and such that the function $x \longmapsto \frac{\partial f}{\partial s}(i,x,s) + \mathbf{v}^{(p)}(i,x) \cdot \nabla f(i,x,s)$ is bounded on $V \times \mathbb{R}_+$. For $f \in \mathcal{D}_H$, we define

$$H^{(p)} f(i,x,s) = \sum_{j} a^{(p)}(i,j,x) \int f(j,y,s)\, \mu^{(p)}_{(i,j,x)}(\mathrm{d}y) + \frac{\partial f}{\partial s}(i,x,s) + \mathbf{v}^{(p)}(i,x) \cdot \nabla f(i,x,s) \quad (1)$$

We now introduce what we called importance functions:

Proposition 2 *Let $t > 0$ and let us assume $\mathcal{H}_1$ to be true. Let us define the function $\varphi_t^{(p)}$ by, for all $(i,x) \in E \times V$:*

$$\varphi_t^{(p)}(i,x,s) = -\int_0^{t-s} \left(P_u^{(p)} h^{(p)}\right)(i,x)\,\mathrm{d}u \text{ if } 0 \leq s \leq t \quad (2)$$

and $\varphi_t^{(p)}(i,x,s) = 0$ otherwise. The function $\varphi_t^{(p)}$ then is the single function element of $\mathcal{D}_H$ solution of the partial differential equation

$$H^{(p)} \varphi_t^{(p)}(i,x,s) = h^{(p)}(i,x)$$

for all $(i,x,s) \in E \times V \times [0,t[$, with initial condition $\varphi_t^{(p)}(i,x,t) = 0$ for all (i,x) in $E \times V$.

The function $\varphi_t^{(p)}$ belongs to $C_2(V \times O)$ and is bounded with all partial derivatives uniformly bounded on $V \times N(p)$ for all $p \in O$.

The function $\varphi_t^{(p)}$ is called the importance function associated to the function $h^{(p)}$ and to t.

The following theorem provides an extension to PDMP of the results from (Gandini 1990).

Theorem 3 *Let $t > 0$ be fixed. Under assumptions $\mathcal{H}_1$, the function $p \longmapsto R_{\rho_0}^{(p)}(t)$ is differentiable with respect of p on $N(p)$ and we have:*

$$\frac{\partial R_{\rho_0}^{(p)}}{\partial p}(t) = \int_0^t \rho_s^{(p)} \frac{\partial h^{(p)}}{\partial p}\,\mathrm{d}s - \int_0^t \rho_s^{(p)} \frac{\partial H^{(p)}}{\partial p} \varphi_t^{(p)}(.,.,s)\,\mathrm{d}s \quad (3)$$

where we set:

$$\frac{\partial H^{(p)}}{\partial p}\varphi(i,x,s) = \sum_{j\in E} \frac{\partial a^{(p)}}{\partial p}(i,j,x) \int \varphi(j,y,s)\, \mu^{(p)}_{(i,j,x)}(\mathrm{d}y) + \sum_{j\in E} a^{(p)}(i,j,x) \frac{\partial}{\partial p}\left(\int \varphi(j,y,s)\, \mu^{(p)}_{(i,j,x)}(\mathrm{d}y)\right) + \frac{\partial v^{(p)}}{\partial p}(i,x) \cdot \nabla\varphi(i,x,s)$$

for all $\varphi \in \mathcal{D}_H$ and all $(i,x,s) \in E \times V \times \mathbb{R}_+$.

Formula (3) is given for one single $p \in \mathbb{R}_+^*$. In case $R_{\rho_0}(t)$ depends on a family of parameters $P = (p_l)_{l\in L}$, we then have:

$$\frac{\partial R_{\rho_0}^{(P)}}{\partial p_l}(t) = \int_0^t \rho_s^{(P)} \frac{\partial h^{(P)}}{\partial p_l}\,\mathrm{d}s - \int_0^t \rho_s^{(P)} \frac{\partial H^{(P)}}{\partial p_l} \varphi_t^{(P)}(.,.,s)\,\mathrm{d}s \quad (4)$$

for all $l \in L$. The numerical assessment of $\frac{\partial R_{\rho_0}^{(P)}}{\partial p_l}(t)$ hence requires the computation of both $\rho_s^{(P)}(i, \mathrm{d}x)$ and $\varphi_t^{(P)}(i,x)$ (independent on $l \in L$). This may be done through two different methods: first, one may use Monte-Carlo simulation to evaluate

$\rho_s^{(P)}(i, \mathrm{d}x)$ and the transition probability distributions $P_t^{(P)}(i, x, j, \mathrm{d}y)$, from where the importance function $\varphi_t^{(P)}(i, x)$ may be derived using (2). Secondly, one may use the finite volume scheme from (Eymard, Mercier, and Prignet 2008), which provides an approximation for $\rho_s^{(P)}(i, \mathrm{d}x)$. The function $\varphi_t^{(P)}(i, x)$ may then be proved to be solution of a dual finite volume scheme, see (Eymard, Mercier, Prignet, and Roussignol 2008). This is the method used in the present paper for the numerical examples provided further. By this method, the computation of $\frac{\partial R_{\rho_0}^{(P)}}{\partial p_l}(t)$ for all $l \in L$ requires the solving of two dual finite volume schemes, as well as some summation for each $l \in L$ involving the data $\frac{\partial h^{(P)}}{\partial p_l}$ and $\frac{\partial H^{(P)}}{\partial p_l}$ (see (4)), which is done simultaneously to the solving.

This has to be compared with the usual finite differences method, for which the evaluation of $\frac{\partial R_{\rho_0}^{(P)}}{\partial p_l}(t)$ for one single p_l requires the computation of $R_{\rho_0}^{(P)}$ for two different families of parameters (P and P with p_l substituted by some $p_l + \varepsilon$). The computation of $\frac{\partial R_{\rho_0}^{(P)}}{\partial p_l}(t)$ for all $l \in L$ by finite differences hence requires $1 + card(L)$ computations. When the number of parameters $card(L)$ is big, the advantage clearly is to the present method.

4 ASYMPTOTIC RESULTS

We are now interested in asymptotic results and we need to assume the process $(I_t, X_t)_{t\geq 0}$ to be uniformly ergodic, according to the following assumptions $\mathcal{H}_2$:

- the process $(I_t, X_t)_{t\geq 0}$ is positive Harris-recurrent with $\pi^{(p)}$ as unique stationary distribution,
- for each $p \in O$, there exists a function $f^{(p)}$ such that $\int_0^{+\infty} f^{(p)}(u)\,\mathrm{d}u < +\infty$, $\int_0^{+\infty} u\, f^{(p)}(u)\,\mathrm{d}u < +\infty$, $\lim_{u\to+\infty} f^{(p)}(u) = 0$ and

$$\left|\left(P_u^{(p)} h^{(p)}\right)(i, x) - \pi^{(p)} h^{(p)}\right| \leq f^{(p)}(u) \qquad (5)$$

  for all $(i, x) \in E \times V$ and all $u \geq 0$.

In order not to give too technical details, we constraint our asymptotic study to the special case where only the jump rates $a^{(p)}(i, j, x)$ and the function $h^{(p)}(i, x)$ depend on the parameter p. Assumptions $\mathcal{H}_1$ are then substituted by **assumptions** $\mathcal{H}_1'$, where conditions on $\mu_{(i,j,x)}$ and on $\mathbf{v}(i, x)$ (now independent on p) are removed.

We may now introduce what we call potential functions.

Proposition 4 *Let us assume $\mu_{(i,j,x)}$ and $\mathbf{v}(i, x)$ to be independent on p and assumptions $\mathcal{H}'_1$, $\mathcal{H}_2$ to be true. Then, the function defined by:*

$$Uh^{(p)}(i, x) := \int_0^{+\infty} \left(\left(P_u^{(p)} h^{(p)}\right)(i, x) - \pi^{(p)} h^{(p)}\right) \mathrm{d}u \qquad (6)$$

exists for all $(i, x) \in E \times V$. Besides, the function $Uh^{(p)}$ is element of $\mathcal{D}_{H_0}$ and it is solution to the ordinary differential equation:

$$H_0^{(p)} Uh^{(p)}(i, x) = \pi^{(p)} h^{(p)} - h^{(p)}(i, x) \qquad (7)$$

for all $(i, x) \in E \times V$. Any other element of $\mathcal{D}_{H_0}$ solution of (7) is of the shape: $Uh^{(p)} + C$ where C is a constant.

The function $Uh^{(p)}$ is called the potential function associated to $h^{(p)}$.

The following theorem provides an extension to PDMP of the results from (Cao and Chen 1997).

Theorem 5 *Let us assume $\mu_{(i,j,x)}$ and $\mathbf{v}(i, x)$ to be independent on p and $\mathcal{H}'_1$, $\mathcal{H}_2$ to be true. Then, the following limit exists and we have:*

$$\lim_{t\to+\infty} \frac{1}{t} \frac{\partial R_{\rho_0}^{(p)}}{\partial p}(t) = \pi^{(p)} \frac{\partial h^{(p)}}{\partial p} + \pi^{(p)} \frac{\partial H_0^{(p)}}{\partial p} Uh^{(p)} \qquad (8)$$

where we set:

$$\frac{\partial H_0^{(p)}}{\partial p} \varphi_0(i, x)$$

$$:= \sum_{j\in E} \frac{\partial a^{(p)}}{\partial p}(i, j, x) \int \varphi_0(j, y)\, \mu_{(i,j,x)}(\mathrm{d}y)$$

for all $\varphi_0 \in \mathcal{D}_{H_0}$ and all $(i, x) \in E \times V$.

Just as for the transitory results, the asymptotic derivative requires, for its numerical assessment, the computation of two different quantities: the asymptotic distribution $\pi^{(p)}(i, \mathrm{d}x)$ and the potential function $Uh^{(p)}(i, x)$. Here again, such computations may be done either by finite volume schemes (using (7) for $Uh^{(p)}$) or by Monte-Carlo simulation (using (6) for $Uh^{(p)}$). Also, in case of a whole set of parameters $P = (p_l)_{l\in L}$, such computations have to be done only once for all $l \in L$, which here again gives the advantage to the present method against finite differences, in case of a large P.

5 A FIRST EXAMPLE

5.1 Presentation - Theoretical results

A single component is considered, which is perfectly and instantaneously repaired at each failure. The distribution of the life length of the component (T_1) is absolutely continuous with respect of Lebesgue measure, with $\mathbb{E}(T_1) > 0$. The successive life lengths make a renewal process. The time evolution of the component is described by the process $(X_t)_{t\geq 0}$ where X_t stands for the time elapsed at time t since the last instantaneous repair (the backward recurrence time). There is one single discrete state so that component I_t is here unnecessary. The failure rate for the component at time t is $\lambda(X_t)$ where $\lambda(\cdot)$ is some non negative function. The process $(X_t)_{t\geq 0}$ is "renewed" after each repair so that $\mu_{(x)}(\mathrm{d}y) = \delta_0(\mathrm{d}y)$ and $(X_t)_{t\geq 0}$ evolves between renewals with speed $\mathbf{v}(x) = 1$.

We are interested in the rate of renewals on $[0,t]$, namely in the quantity $Q(t)$ such that:

$$Q(t) = \frac{R(t)}{t} = \frac{1}{t}\mathbb{E}_0\left(\int_0^t \lambda(X_s)\ \mathrm{d}s\right)$$

where $R(t)$ is the renewal function associated to the underlying renewal process.

The function $\lambda(x)$ is assumed to depend on some parameter $p > 0$.

Assuming $\lambda(x)$ to meet with $\mathcal{H}_1'$ requirement, the results from Section 3 here writes:

$$\frac{\partial Q^{(p)}(t)}{\partial p} = \frac{1}{t}\int_0^t\int_0^s \rho_s^{(p)}(\mathrm{d}x)\frac{\partial\lambda^{(p)}}{\partial p}(x) \times\left(1-\varphi_t^{(p)}(0,s)+\varphi_t^{(p)}(x,s)\right)\ \mathrm{d}s$$

where φ_t is solution of

$$\lambda(x)(\varphi_t(0,s)-\varphi_t(x,s)) + \frac{\partial}{\partial s}\varphi_t(x,s)+\frac{\partial}{\partial x}\varphi_t(x,s)=\lambda(x)$$

for all $s\in[0,t[$ and $\varphi_t(x,t)=0$ for all $x\in[0,t]$.

Assuming $\mathbb{E}(T_1) < +\infty$, the process is known to have a single stationary distribution $\pi^{(p)}$ which has the following probability density function (p.d.f.):

$$f_\pi^{(p)}(x) = \frac{\mathbb{P}\left(T_1^{(p)} > x\right)}{\mathbb{E}\left(T_1^{(p)}\right)} = \frac{e^{-\int_0^x \lambda^{(p)}(u)\mathrm{d}u}}{\mathbb{E}\left(T_1^{(p)}\right)} \tag{9}$$

Using a result from (Konstantopoulos and Last 1999), one may then prove the following proposition, which ensure the process to be uniformly ergodic, meeting with $\mathcal{H}_2$:

Proposition 6 *Let us assume that $\mathbb{E}\left(e^{\delta T_1}\right) < +\infty$ for some $0<\delta<1$ and that T_1 is new better than used (NBU: for all $x,t\geq 0$, we have $\bar{F}(x+t)\leq\bar{F}(x)\bar{F}(t)$, where $\bar{F}$ is the survival function $\bar{F}(t)=\mathbb{P}(T_1>t)$). Then, there are some $C<+\infty$ and $0<\rho<1$ such that:*

$$\left|P_t^{(p)}h^{(p)}(x)-\pi^{(p)}h^{(p)}\right|\leq C\rho^t$$

for all $x\in\mathbb{R}_+$.

Under assumption $\mathcal{H}_2$, we get the following closed form for $\frac{\partial Q(\infty)}{\partial p}$:

$$\frac{\partial Q(\infty)}{\partial p} = \frac{1}{\mathbb{E}_0(T_1)}\int_0^{+\infty}\frac{\partial\lambda}{\partial p}(x) \times\left(1-Q(\infty)\int_0^x e^{-\int_0^v\lambda(u)\mathrm{d}u}\mathrm{d}v\right)\mathrm{d}x$$

5.2 Numerical results

We assume that T_1 is distributed according to some Weibull distribution, which is slightly modified to meet with our assumptions:

$$\lambda^{(\alpha,\beta)}(x) = \begin{cases} \alpha\beta x^{\beta-1} \text{ if } x<x_0 \\ P_{\alpha,\beta,x_0}(x) \text{ if } x_0\leq x<x_0+2 \\ \alpha\beta(x_0+1)^{\beta-1} = \text{constant if } x_0+2\leq x\end{cases}$$

where $(\alpha,\beta)\in O =]0,+\infty[\times]2,+\infty[$, x_0 is chosen such that $T_1>x_0$ is a rare event ($\mathbb{P}_0(T_1>x_0)=e^{-\alpha x_0^\beta}$ small) and $P_{\alpha,\beta,x_0}(x)$ is some smoothing function which makes $x\longmapsto\lambda^{(\alpha,\beta)}(x)$ continuous on $\mathbb{R}_+$. For such a failure rate, it is then easy to check that assumptions $\mathcal{H}_1'$ and $\mathcal{H}_2$ are true, using Proposition 6.

Taking $(\alpha,\beta)=(10^{-5},4)$ and $x_0=100$ (which ensures $\mathbb{P}_0(T_1>x_0)\simeq 5\times10^{-435}$), we are now able to compute $IF_\alpha(t)$ and $IF_\beta(\infty)$ for $t\leq\infty$. In order to validate our results, we also compute such quantities by finite differences (FD) using:

$$\frac{\partial Q(t)}{\partial p}\simeq\frac{1}{\varepsilon}\left(Q^{(p+\varepsilon)}(t)-Q^{(p)}(t)\right)$$

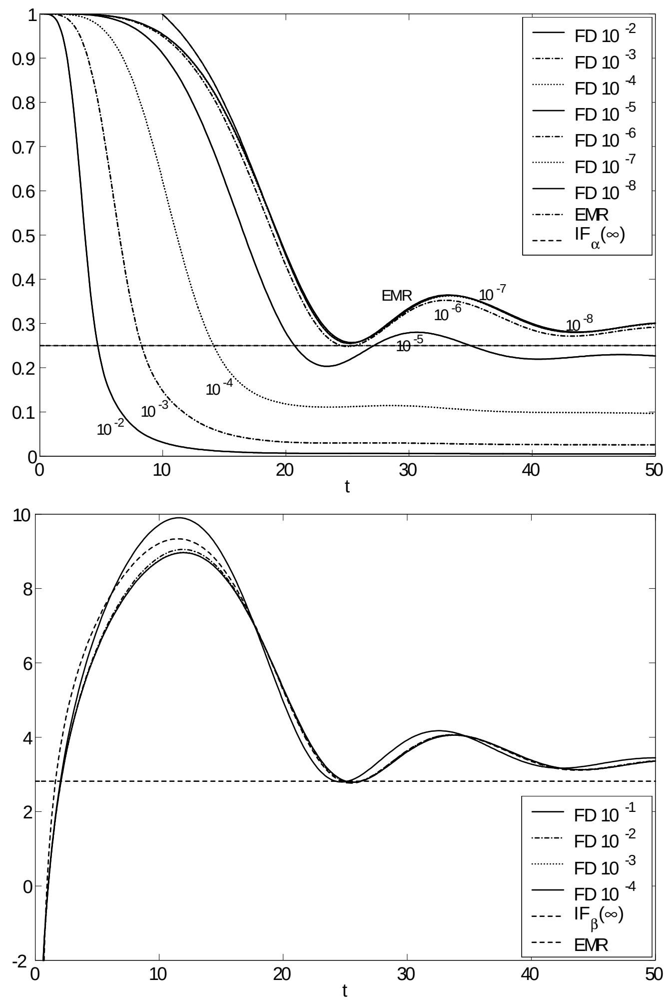
for small ε and $t\leq\infty$. For the transitory results, we use the algorithm from (Mercier 2007) which provides an estimate for the renewal function $R^{(p)}(t)$ and hence for $Q^{(p)}(t)=\frac{R^{(p)}(t)}{t}$ to compute $Q^{(p)}(t)$ and $Q^{(p+\varepsilon)}(t)$. For the asymptotic results, we use the exact formula $Q^{(p)}(\infty)=\frac{1}{\mathbb{E}_0\left(T_1^{(p)}\right)}$ to compute such quantities, which is a direct consequence of the key renewal theorem.

The results are gathered in Table 1 for the asymptotic importance factors $IF_p(\infty)$.

Table 1: $IF_\alpha(\infty)$ and $IF_\beta(\infty)$ by finite differences (FD) and by the present method (EMR)

	ε	$IF_\alpha(\infty)$	$IF_\beta(\infty)$
FD	10^{-2}	4.625×10^{-3}	2.824
	10^{-4}	8.212×10^{-2}	2.821
	10^{-6}	2.411×10^{-1}	2.821
	10^{-8}	2.499×10^{-1}	2.821
	10^{-10}	2.500×10^{-1}	2.821
EMR	-	2.500×10^{-1}	2.821

The results are very stable for $IF_\beta(\infty)$ by FD choosing different values for ε and FD give very similar results as EMR. The approximation for $IF_\alpha(\infty)$ by FD requires smaller ε to give similar results as EMR. Similar remarks are valid for the transitory results, which are plotted in Figures 1 and 2 for $t \in [0, 50]$ and different values of ε. This clearly validates the method. As for the results, we may note that, for a Weibull distribution, the shape parameter β is much more influent on the rate of renewals than the scale parameter α.



Figures 1 & 2. $IF_\alpha(t)$ and $IF_\beta(t)$ by finite differences and by the present method (EMR)

6 A second example

6.1 Presentation - Theoretical results

The following example is very similar to that from (Boxma, Kaspi, Kella, and Perry 2005). The main difference is that we here assume X_t to remain bounded ($X_t \in [0, R]$) whereas, in the quoted paper, X_t takes its values in $\mathbb{R}_+$.

A tank is considered, which may be filled in or emptied out using a pump. This pump may be in two different states: "in" (state 0) or "out" (state 1). The level of liquid in the tank goes from 0 up to R. The state of the system "tank-pump" at time t is (I_t, X_t) where I_t is the discrete state of the pump ($I_t \in \{0, 1\}$) and X_t is the continuous level in the tank ($X_t \in [0, R]$). The transition rate from state 0 (resp. 1) to state 1 (resp. 0) at time t is $\lambda_0(X_t)$ (resp. $\lambda_1(X_t)$). The speed of variation for the liquid level in state 0 is $\mathbf{v}_0(x) = r_0(x)$ with $r_0(x) > 0$ for all $x \in [0, R[$ and $r_0(R) = 0$: the level increases in state 0 and tends towards R. Similarly, the speed in state 1 is $\mathbf{v}_1(x) = -r_1(x)$ with $r_1(x) > 0$ for all $x \in]0, R]$ and $r_1(0) = 0$: the level of liquid decreases in state 1 and tends towards 0. For $i = 0, 1$, the function λ_i (respectively r_i) is assumed to be continuous (respectively Lipschitz continuous) and consequently bounded on $[0, R]$. The level in the tank is continuous so that $\mu(i, 1-i, x)(\mathrm{d}y) = \delta_x(\mathrm{d}y)$ for $i \in \{0, 1\}$, all $x \in [0, R]$. In order to ensure the process to be positive Harris recurrent, we also make the following additional assumptions: $\lambda_1(0) > 0$, $\lambda_0(R) > 0$ and

$$\int_x^R \frac{1}{r_0(u)} \mathrm{d}u = +\infty, \quad \int_0^y \frac{1}{r_1(u)} \mathrm{d}u = +\infty$$

for all $x, y, \in]0, R[$. We get the following result:

Proposition 7 *Under the previous assumptions, the process $(I_t, X_t)_{t\geq 0}$ is positive Harris recurrent with single invariant distribution π given by:*

$$\pi(i, \mathrm{d}x) = f_i(x)\,\mathrm{d}x$$

for $i = 0, 1$ and

$$f_0(x) = \frac{K_\pi}{r_0(x)} e^{\int_{R/2}^x \left(\frac{\lambda_1(u)}{r_1(u)} - \frac{\lambda_0(u)}{r_0(u)}\right)\mathrm{d}u} \qquad (10)$$

$$f_1(x) = \frac{K_\pi}{r_1(x)} e^{\int_{R/2}^x \left(\frac{\lambda_1(u)}{r_1(u)} - \frac{\lambda_0(u)}{r_0(u)}\right)\mathrm{d}u} \qquad (11)$$

where $K_\pi > 0$ is a normalization constant. Besides, assumptions $\mathcal{H}_2$ are true, namely, the process $(I_t, X_t)_{t\geq 0}$ is uniformly ergodic.

6.2 Quantities of interest

We are interested in two quantities: first, the proportion of time spent by the level in the tank between two fixed bounds $\frac{R}{2}-a$ and $\frac{R}{2}+b$ with $0<a,b<\frac{R}{2}$ and we set:

$$Q_1(t) = \frac{1}{t}\mathbb{E}_{\rho_0}\left(\int_0^t \mathbf{1}_{\left\{\frac{R}{2}-a\le X_s\le \frac{R}{2}+b\right\}}\mathrm{d}s\right)$$

$$= \frac{1}{t}\sum_{i=0}^{1}\int_0^t\int_{\frac{R}{2}-a}^{\frac{R}{2}+b}\rho_s(i,\mathrm{d}x)\ \mathrm{d}s$$

$$= \frac{1}{t}\int_0^t \rho_s h_1\ \mathrm{d}s \tag{12}$$

with $h_1(i,x) = \mathbf{1}_{\left[\frac{R}{2}-a,\frac{R}{2}+b\right]}(x)$.

The second quantity of interest is the mean number of times the pump is turned off, namely turned from state "in" (0) to state "out" (1) by unit time, namely:

$$Q_2(t) = \frac{1}{t}\mathbb{E}_{\rho_0}\left(\sum_{0<s\le t}\mathbf{1}_{\{I_{s-}=0 \text{ and } I_s=1\}}\right)$$

$$= \frac{1}{t}\mathbb{E}_{\rho_0}\left(\int_0^t \lambda_0(X_s)\mathbf{1}_{\{I_s=0\}}\mathrm{d}s\right)$$

$$= \frac{1}{t}\int_0^t\int_0^R \lambda_0(x)\rho_s(0,\mathrm{d}x)\ \mathrm{d}s$$

$$= \frac{1}{t}\int_0^t \rho_s h_2\ \mathrm{d}s \tag{13}$$

with $h_2(i,x) = \mathbf{1}_{\{i=0\}}\lambda_0(x)$.

For $i=0,1$, the function $\lambda_i(x)$ is assumed to depend on some parameter α_i (but no other data depends on the same parameter). Similarly, the function $r_i(x)$ depends on some ρ_i for $i=0,1$. By definition, the function h_1 also depends on parameters a and b.

We want to compute the importance factors with respect to p for $p\in\{\alpha_0,\alpha_1,r_0,r_1,a,b\}$ both in Q_1 and Q_2, except for parameters a and b which intervenes only in Q_1.

As told at the end of Section 13, we have to compute the marginal distribution $(\rho_s(i,\mathrm{d}x))_{i=0,1}$ for $0\le s\le t$ and the importance function associated to h_{i_0} and t for $i_0=1,2$. This is done through solving two dual implicit finite volume schemes. A simple summation associated to each p, which is done simultaneously to the solving, then provides the result through (4).

As for the asymptotic results, the potential functions Uh_{i_0} are here solutions of

$$v_i(x)\frac{d}{dx}\left(Uh_{i_0}(i,x)\right)$$

$$+\lambda_i(x)\left(Uh_{i_0}(1-i,x) - Uh_{i_0}(i,x)\right)$$

$$= Q_{i_0}(\infty) - h_{i_0}(i,x)$$

for $i_0=0,1$, which may be solved analytically. A closed form is hence available for $\frac{\partial Q_{i_0}(\infty)}{\partial p}$ using $(10-11)$ and (8).

6.3 Numerical example

We assume the system to be initially in state $(I_0,X_0)=(0,R/2)$. Besides, we take:

$$\lambda_0(x) = x^{\alpha_0}\ ;\ r_0(x) = (R-x)^{\rho_0}\ ;$$

$$\lambda_1(x) = (R-x)^{\alpha_1}\ ;\ r_1(x) = x^{\rho_1}$$

for $x\in[0,R]$ with $\alpha_i>1$ and $\rho_i>1$. All conditions for irreducibility are here achieved.

We take the following numerical values:

$$\alpha_0 = 1.05; \rho_0 = 1.2; \alpha_1 = 1.10;$$

$$\rho_1 = 1.1; R = 1; a = 0.2; b = 0.2.$$

Similarly as for the first method, we test our results using finite differences (FD). The results are here rather stable choosing different values for ε and the results are provided for $\varepsilon=10^{-2}$ in case $p\in\{\alpha_0,\alpha_1,r_0,r_1\}$ and for $\varepsilon=10^{-3}$ in case $p\in\{a,b\}$. The asymptotic results are given in Tables 2 and 3, and the transitory ones are given in Table 4 and 5 for $t=2$.

Table 2: $IF_p^{(1)}(\infty)$ by the present method (EMR) and by finite differences (FD)

p	FD	EMR	Relative error
α_0	-3.59×10^{-2}	-3.57×10^{-2}	$5{,}40\times10^{-3}$
α_1	-4.45×10^{-2}	-4.43×10^{-2}	$3{,}65\times10^{-3}$
ρ_0	3.19×10^{-1}	3.17×10^{-1}	$6{,}95\times10^{-3}$
ρ_1	2.80×10^{-1}	2.78×10^{-1}	$7{,}19\times10^{-3}$
a	4.98×10^{-1}	4.98×10^{-1}	$1{,}06\times10^{-7}$
b	5.09×10^{-1}	5.09×10^{-1}	$1{,}53\times10^{-7}$

Table 3: $IF_p^{(2)}(\infty)$ by the present method (EMR) and by finite differences (FD)

p	FD	EMR	Relative error
α_0	-1.81×10^{-1}	-1.81×10^{-1}	$1{,}67\times10^{-4}$
α_1	-1.71×10^{-1}	-1.71×10^{-1}	$1{,}30\times10^{-4}$
ρ_0	-6.22×10^{-2}	-6.19×10^{-2}	$5{,}21\times10^{-3}$
ρ_1	-6.05×10^{-2}	-6.01×10^{-2}	$5{,}58\times10^{-3}$

Table 4: $IF_p^{(1)}(t)$ for $t = 2$ by the present method (EMR) and by finite differences (FD)

p	FD	EMR	Relative error
α_0	-8.83×10^{-2}	-8.82×10^{-2}	$1,08 \times 10^{-3}$
α_1	-9.10×10^{-3}	-9.05×10^{-3}	$5,29 \times 10^{-3}$
ρ_0	4.89×10^{-1}	4.85×10^{-1}	$7,51 \times 10^{-3}$
ρ_1	1.97×10^{-1}	1.97×10^{-1}	$4,04 \times 10^{-3}$
a	2.48×10^{-1}	2.48×10^{-1}	$4,89 \times 10^{-4}$
b	7.11×10^{-1}	7.11×10^{-1}	$7,77 \times 10^{-6}$

Table 5: $IF_p^{(2)}(t)$ for $t = 2$ by the present method (EMR) and by finite differences (FD)

p	FD	EMR	Relative error
α_0	-2.06×10^{-1}	-2.06×10^{-1}	$9,12 \times 10^{-4}$
α_1	-6.80×10^{-2}	-6.79×10^{-2}	$2,12 \times 10^{-3}$
ρ_0	-1.25×10^{-1}	-1.24×10^{-1}	$4,27 \times 10^{-3}$
ρ_1	-4.11×10^{-3}	-4.03×10^{-3}	$2,00 \times 10^{-2}$

The results are very similar by FD and MR both for the asymptotic and transitory quantities, which clearly validate the method. Note that the asymptotic results coincides by both methods, even in the case where the velocity $\mathbf{v}(i, x)$ field depends on the parameter (here ρ_i), which however does not fit with our technical assumptions from Section 4. Due to that (and to other examples where the same remark is valid), one may conjecture that the results from Section 4 are valid under less restrictive assumptions than those given in that section.

As for the results, one may note that the importance factors at $t = 2$ of α_0 and ρ_0 in Q_i $(i = 1, 2)$ are clearly higher than the importance factors of α_1 and ρ_1 in Q_i $(i = 1, 2)$. This must be due to the fact that the system starts from state 0, so that on $[0, 2]$, the system spends more time in state 0 than in state 1. The parameters linked to state 0 hence are more important than the ones linked to state 1. Similarly, the level is increasing in state 0 so that the upper bound b is more important than the lower one a.

In long-time run, the importance factors of α_0 and α_1 in Q_i $(i = 1, 2)$ are comparable. The same remark is valid for ρ_0 and ρ_1, as well as for a and b.

Finally, parameters ρ_0 and ρ_1 are more important than parameters α_0 and α_1 in Q_1, conversely to what happens in Q_2. This seems coherent with the fact that quantity Q_1 is linked to the level in the tank, and consequently to its evolution, controlled by ρ_0 and ρ_1, whereas quantity Q_2 is linked to the transition rates, and consequently to α_0 and α_1.

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REFERENCES

Boxma, O., H. Kaspi, O. Kella, and D. Perry (2005). On/off storage systems with state-dependent input, output, and switching rates. *Probab. Engrg. Inform. Sci. 19*(1), 1–14.

Cao, X.-R. and H.-F. Chen (1997). Perturbation realization, potentials, and sensitivity analysis ofMarkov processes. *IEEE trans. automat. contr. 42*(10), 1382 – 1393.

Cocozza-Thivent, C., R. Eymard, S. Mercier, and M. Roussignol (2006). Characterization of the marginal distributions of Markov processes used in dynamic reliability. *J. Appl. Math. Stoch. Anal.*, Art. ID 92156, 18.

Davis, M. H. A. (1984). Piecewise-deterministic Markov processes: a general class of nondiffusion stochastic models. *J. Roy. Statist. Soc. Ser. B 46*(3), 353–388. With discussion.

Devooght, J. (1997). Dynamic reliability. *Advances in Nuclear Science and Technology 25*, 215–278.

Eymard, R., S. Mercier, and A. Prignet (2008). An implicit finite volume scheme for a scalar hyperbolic problem with measure data related to piecewise deterministic markov processes. *J. Comput. Appl. Math. available online 1 Nov. 2007*.

Eymard, R., S. Mercier, A. Prignet, and M. Roussignol (2008, jun). A finite volume scheme for sensitivity analysis in dynamic reliability. In *Finite Volumes for Complex Applications V*, Aussois, France.

Gandini, A. (1990). Importance & sensitivity analysis in assessing system reliability. *IEEE trans. reliab. 39*(1), 61–70.

Konstantopoulos, T. and G. Last (1999). On the use of Lyapunov function methods in renewal theory. *Stochastic Process. Appl. 79*(1), 165–178.

Lions, J.-L. (1968). *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*. Paris: Dunod.

Mercier, S. (2007). Discrete random bounds for general random variables and applications to reliability. *European J. Oper. Res. 177*(1), 378–405.

Mercier, S. and M. Roussignol (2007). Sensitivity estimates in dynamic reliability. In *Proceedings of MMR 2007 (Fifth International Conference on Mathematical Methods on Reliability)*, Glasgow, Scotland.