Multivariate Degradation Model with dependence due to Shocks

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The aim of this paper is to model degradation phenomena in a multi-unit context taking into account stochastic dependence between the units. Here, we intend to propose a model structure that allows in the same time enough complexity for the representation of phenomena and enough simplicity for the analytical calculations. More precisely, a multi-unit system is considered, which is submitted to a random stressing environment which arrives by shocks. The model takes into account two types of dependence between the components: firstly, a shock impacts all components at the same time; secondly, for a given shock, the deterioration increments of the different components are considered to be correlated. The intrinsic deterioration of the n (say) units is modeled through independent stochastic processes $(Z_t^{(i)})_{t\geq 0}$, with $1 \leq i \leq n$. Given the usual nature of the degradation phenomena, is it seems reasonable to suppose that each $(Z_t^{(i)})_{t\geq 0}$ should be a monotone process with continuous state space. Accordingly, the shocks are classically assumed to arrive independently, according to a Poisson process $(N_t)_{t\geq 0}$. The parameter estimation (moment method) and the reliability assessment are presented for any multi-unit systems with coherent structure. At last a numerical results is presented with a 3 units system.

Keywords: Multivariate Lévy process, poisson process, shock model, prognostic, reliability, stochastic dependence

1 Introduction

The aim of this paper is to model multivariate degradation phenomena in a multi-unit context taking into account stochastic dependence between the units. This kind of model is not so widely studied in the framework of reliability assessment, prognostics and maintenance optimization since the mathematical developments associated to such cases lead to cumbersome numerical computations. Here, we intend to propose a model structure that allows in the same time enough complexity for the representation of phenomena and enough simplicity for the analytical calculations. Starting from given multivariate stochastic processes with given stochastic dependence, we investigate which quantities are tractable and show that we have a relevant set of analytical solutions for both the estimation step and the reliability assessment step. As far as we know this precise structure has not been studied previously in such a context and gives enough latitude to model a large class of realistic situations. Hence it turns to be a good candidate to model stochastic dependence and degradation phenomena in a systems engineering context, especially for reliability assessment and prognostics.

More precisely, a multi-unit system is considered, which is submitted to a random stressing environment which arrives by shocks. These shocks may be due to some external specific demand, some significative change of the operational condition, of the environments, etc... These shocks simultaneously affect all the components of the system and make them dependent. Without any shock, the intrinsic deteriorations of all components are independent. A shock is assumed to increase the deterioration of each component by a random amount. The model takes into account two types of dependence between the components: firstly,

a shock impacts all components at the same time; secondly, for a given shock, the deterioration increments of the different components are considered to be correlated.
The intrinsic deterioration of the n (say) units is modeled through independent stochastic processes $\left(Z_t^{(i)}\right)_{t\geq 0}$, with $1 \leq i \leq n$. Given the usual nature of the degradation phenomena, is it seems reasonable to suppose that each $\left(Z_t^{(i)}\right)_{t\geq 0}$ should be a monotone process with continuous stat space. Accordingly, the shocks are classically assumed to arrive independently, according to a Poisson process $(N_t)_{t\geq 0}$. By a shock, the deterioration of each component is increased. This is modeled assuming that all components are aging at once at each shock arrival of a random amount, say $U_j^{(i)}$ for the $j-$th shock and for the $i-$th component. The deterioration level of the $i-$th component at time t hence is

$$X_t^{(i)} = Z_{t+Y_t^{(i)}}^{(i)}$$

where $Y_t^{(i)} = \sum_{j=1}^{N_t} U_j^{(i)}$ stands for the accumulated jumps in the age of $i-$th component due to the shocks. This leads to a multivariate model $X_t = \left(X_t^{(1)}, \cdots, X_t^{(n)}\right)$, where all the dependence between the $X_t^{(i)}$'s is induced by $Y_t = \left(Y_t^{(1)}, \cdots, Y_t^{(n)}\right)$, namely by the shocks. Note that process $(X_t)_{t\geq 0}$ is obtained through

$$X_t = \left(Z_{t+Y_t^{(1)}}^{(1)}, \cdots, Z_{t+Y_t^{(n)}}^{(n)}\right), \tag{1}$$

so that the model corresponds to some multivariate time change in $Z_t = \left(Z_t^{(1)}, \cdots, Z_t^{(n)}\right)$.
Due to its tractability and its ability to fit a large kind of data sets in the context of monotone paths, a univariate Gamma process has been chosen for each $\left(Z_t^{(i)}\right)_{t\geq 0}$. We will see in the following that time changes at shock times is just equivalent to some sudden change in the shape parameters of the initial underlying Gamma processes. Between jumps, the shape parameters evolve linearly with time, just as for ordinary Gamma processes. This means that between jumps, the components evolve independently as homogeneous Gamma processes. From a practical point of view, it means that we have n degradation phenomena whose speed is not changed by the shocks, but whose level can be significantly increased at each shock event.

The remaining of the paper is organized as follows: in section 2, the model is presented and the Laplace transform of the degradation process is given. In section 3 an estimation scheme is proposed in case of periodic inspections based on a method of moments. In section 4, the reliability is calculated for any coherent system structure and the influence of the shock model parameters is studied. The results of sections 3 and 4 are illustrated by numerical examples in a case of three-unit system.

2 Degradation modeling

2.1 Definition of the model

Let us consider the following hypotheses and notations:

- $Y = (Y_t)_{t\geq 0}$: a compound Poisson process, in $\mathbb{R}_+^n$, with

$$Y_t = \sum_{j=1}^{N_t} U_j = \left(Y_t^{(1)}, \cdots, Y_t^{(n)}\right) \text{ and } Y_t^{(i)} = \sum_{j=1}^{N_t} U_j^{(i)}$$

  where:

  - $U_j = \left(U_j^{(1)}, , \cdots, U_j^{(n)}\right)$,

- U_1, ..., U_j, ... : i.i.d. independent from $(N_t)_{t\geq 0}$,
- $U = \left(U^{(1)}, \cdots, U^{(n)}\right)$ is a generic version of U_j with distribution ν,
- $N = (N_t)_{t\geq 0}$: a Poisson process with parameter λ.

- $Z_t = \left(Z_t^{(1)}, \cdots, Z_t^{(n)}\right)_{t\geq 0}$, where $\left(Z_t^{(i)}\right)_{t\geq 0}$ are univariate independent Gamma processes with the same distribution. With no loss of generality, $\forall \leq s < t$ and $\forall i \in \{1, \ldots, n\}$, $Z_t^{(i)} - Z_s^{(i)}$ is supposed to be gamma-distributed with shape $t - s$ and scale 1. Hence the intrinsic deterioration can be restated as $Z_{at} = \left(Z_{a_1 t}, \ldots, Z_{a_2 t}\right)$ where $a = (a_1, \cdots, a_n)$.

- Y and Z are supposed to be independent.

Then we can define the whole degradation process X_t as follows:

$$X_t = Z_{at+Y_t} = \left(Z^{(1)}_{a_1 t + Y_t^{(1)}}, \cdots, Z^{(n)}_{a_n t + Y_t^{(n)}}\right), \text{ with } a = (a_1, \cdots, a_n).$$

We can also see X_t as the sum of two independent processes: $X_t = X_t^{\perp} + X_t^{\parallel}$ where $X_t^{\perp} = \left(X_t^{(1)\perp}, ..., X_t^{(n)\perp}\right)$, and $X_t^{\parallel} = \left(X_t^{(1)\parallel}, ..., X_t^{(n)\parallel}\right)$, with $X_t^{(i)\perp} \overset{d}{=} Z^{(i)}_{a_i t}$ and $X_t^{(i)\parallel} \overset{d}{=} Z^{(i)}_{Y_t^{(i)}}$. where $\overset{d}{=}$ stands for the equality in distribution. $X^{\perp}$ consists in n independent Gamma processes with parameters $(a_i, 1)$. The whole dependence between the components of X is included in the dependence between the components of $X^{\parallel}$.

We should notice here that since gamma processes are Lévy processes, $(Z_t)_{t\geq 0}$ is the conjonction of n independent Lévy process and hence is a multivariate Lévy process. Also, Y_t is a multivariate compound Poisson process and hence is a non decreasing Lévy process. Based on (1), the process $(X_t)_{t\geq 0}$ is obtained through multivariate subordination of the Lévy process $(Z_t)_{t\geq 0}$ and hence is a (multivariate) Lévy process.

2.2 Laplace transform

For $x = (x_1, \cdots, x_n), y = (y_1, , \cdots, y_n) \in \mathbb{R}^n$, let's note $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. The Laplace transform can be calculated analytically and is given by the following proposition.

Proposition 2.1 *For $s = (s_1, \cdots, s_n) \in \mathbb{R}_+^n$, we get*

$$\mathcal{L}_{X_t}(s) = \mathbb{E}\left(e^{-\langle s, X_t\rangle}\right) = \prod_{i=1}^{n} (1 + s_i)^{-a_i t} \ \ e^{-\lambda t \ (1 - \mathcal{L}_U(\ln(1+s)))}$$

with

$$\mathcal{L}_U(\ln(1+s)) = \mathbb{E}\left(\prod_{j=1}^{n} (1 + s_j)^{-U^{(j)}}\right).$$

Using the Laplace transform, it is possible to show that the process is theoretically identifiable.

3 Estimation

3.1 Moments

Using the Laplace transform, the moments can also be calculated and are given by the following proposition.

Proposition 3.1

$$\mathbb{E}\left(X_t^{(i)}\right) = t\ m^{(i)} \text{ with } m^{(i)} = a_i + \lambda\ \mathbb{E}\left(U_1^{(i)}\right)$$
$$var\left(X_t^{(i)}\right) = t\ \mu_2^{(i)} \text{ with } \mu_2^{(i)} = a_i + \lambda\ \mathbb{E}\left(U^{(i)}\right) + \lambda\ \mathbb{E}\left(U^{(i)2}\right) = m^{(i)} + \lambda\ \mathbb{E}\left(U^{(i)2}\right)$$
$$cov\left(X_t^{(i)}, X_t^{(j)}\right) = t\ c_{i,j} \text{ with } c_{i,j} = \lambda\ \mathbb{E}\left(U^{(i)}U^{(j)}\right)$$
$$\mathbb{E}\left(\left(X_t^{(i)} - \mathbb{E}\left(X_t^{(i)}\right)\right)^3\right) = t\ \mu_3^{(i)} \text{ with } \mu_3^{(i)} = 2a_i + \lambda\ \mathbb{E}\left(U^{(i)}\left(U^{(i)}+1\right)\left(U^{(i)}+2\right)\right)$$

3.2 Estimation in case of periodic inspections

Consider M paths which are periodically and perfectly inspected every Δt unit of time. Multivariate observations $x_{m,j} = \left(x_{m,j}^1, \ldots, x_{m,j}^n\right)$ are available where m is the path number $m \in \{1, \ldots M\}$, j stands for the inspection time $t_j = j\Delta t$ and $x_{m,j}$ is an observation of X_{t_j}. From the observed degradation levels a collection of degradation increments $(\Delta x_i)_{1\le i\le N}$ is obtained for the random vector$(\Delta X_i)_{1\le i\le N}$, with $\Delta X_i = X_{i\Delta t} - X_{(i-1)\Delta t}$. Let's note $m^{(j)} = \mathbb{E}\left(X_1^{(j)}\right)$, $v^{(j)} = var\left(X_1^{(j)}\right)$, $c_{j,k} = cov\left(X_1^{(j)}, X_1^{(k)}\right)$. The following unbiaised estimators can be obtained [1], [3]:

$$\hat{m}^{(j)} = \frac{\sum_{k=1}^N \Delta X_k^{(j)}}{N\Delta t} \quad , \quad \hat{v}^{(j)} = \frac{\sum_{k=1}^N \left(\Delta X_k^{(j)} - \hat{m}^{(j)}\ \Delta t\right)^2}{(N-1)\Delta t}, \hat{c}_{j,k} = \frac{\sum_{i=1}^N \left(\Delta X_i^{(j)} - \hat{m}_j\ \Delta t\right)\left(\Delta X_i^{(k)} - \hat{m}^{(k)}\ \Delta t\right)}{(N-1)\Delta t}. \tag{2}$$

An additional equation may be required for the estimation, so we propose an unbiased estimator of the third-order moment $\mu_3^{(j)}$.

Proposition 3.2 *Let's note* $\hat{\mu}_3^{(i)} = \frac{\sum_{k=1}^N \left(X_k^{(i)} - \hat{m}^{(i)}\ \Delta t\right)^3}{\Delta t\ \left(N-3+\frac{4}{N}\right)}$ *then* $\mathbb{E}\left(\hat{\mu}_3^{(i)}\right) = \mu_3^{(i)}$.

3.3 Numerical Illustration

Consider a special case with:

- $V^{(i)}$, $i = 1, 2, 3$ are independent and their law is $\mathcal{E}(\lambda_i)$,
- $U^{(1)} = V^{(1)} + V^{(3)}$, $U^{(2)} = V^{(2)} + V^{(3)}$.

For $i \in \{1, 2\}$, we get:

$$m^{(i)} = a_i + \lambda\ \mathbb{E}\left(U^{(i)}\right), \mu_2^{(i)} - m^{(i)} = \lambda\ \mathbb{E}\left(U^{(i)2}\right), c_{1,2} = \lambda\ \mathbb{E}\left(U^{(1)}U^{(2)}\right),$$
$$\mu_3^{(i)} = 2a_i + \lambda\ \mathbb{E}\left(U^{(i)}\left(U^{(i)}+1\right)\left(U^{(i)}+2\right)\right) = 2a_i + \lambda\ \mathbb{E}\left(U^{(i)3}\right) + 3\lambda\ \mathbb{E}\left(U^{(i)2}\right) + 2\lambda\ \mathbb{E}\left(U^{(i)}\right)$$

so: $\mu_3^{(i)} = 2a_i + \lambda\ \mathbb{E}\left(U^{(i)3}\right) + 3\lambda\ \mathbb{E}\left(U^{(i)2}\right) + 2\lambda\ \mathbb{E}\left(U^{(i)}\right) = \lambda\ \mathbb{E}\left(U^{(i)3}\right) + 3\mu_2^{(i)} - m^{(i)}$

The following equations are obtained for $i = 1, 2:$ $m^{(i)} = a_i + \lambda\ \mathbb{E}\left(U^{(i)}\right), \mu_2^{(i)} - m^{(i)} = \lambda\ \mathbb{E}\left(U^{(i)2}\right), \mu_3^{(i)} - 3\mu_2^{(i)} + m^{(i)} = \lambda\ \mathbb{E}\left(U^{(i)3}\right)$, and we have also: $c_{1,2} = \lambda\ \mathbb{E}\left(U^{(1)}U^{(2)}\right)$.

Then, we get :

$$\mathbb{E}\left(U^{(i)}\right) = \mathbb{E}\left(V^{(i)}\right) + \mathbb{E}\left(V^{(3)}\right),$$
$$\mathbb{E}\left(U^{(i)2}\right) = \mathbb{E}\left(\left(V^{(i)}\right)^2\right) + 2\ \mathbb{E}\left(V^{(i)}\right)\mathbb{E}\left(V^{(3)}\right) + \mathbb{E}\left(\left(V^{(3)}\right)^2\right),$$
$$\mathbb{E}\left(U^{(i)3}\right) = \mathbb{E}\left(\left(V^{(i)}\right)^3\right) + 3\ \mathbb{E}\left(V^{(i)2}\right)\mathbb{E}\left(V^{(3)}\right) + 3\ \mathbb{E}\left(V^{(i)}\right)\mathbb{E}\left(V^{(3)2}\right) + \mathbb{E}\left(\left(V^{(3)}\right)^3\right),$$
$$\mathbb{E}\left(U^{(1)}U^{(2)}\right) = \mathbb{E}\left(V^{(1)}\right)\mathbb{E}\left(V^{(3)}\right) + \mathbb{E}\left(V^{(2)}\right)\mathbb{E}\left(V^{(3)}\right) + \mathbb{E}\left(V^{(3)2}\right)$$

Moreover for an exponential law we have $\mathbb{E}\left(V^{(i)n}\right)=\frac{n!}{\lambda_i^n}$. So:

$$\mathbb{E}\left(U^{(i)}\right)=\frac{1}{\lambda_i}+\frac{1}{\lambda_3},\mathbb{E}\left(U^{(i)2}\right)=2\left(\frac{1}{\lambda_i^2}+\frac{1}{\lambda_i\lambda_3}+\frac{1}{\lambda_3^2}\right),$$
$$\mathbb{E}\left(U^{(i)3}\right)=6\left(\frac{1}{\lambda_i^3}+\frac{1}{\lambda_i^2\lambda_3}+\frac{1}{\lambda_3^2\lambda_i}+\frac{1}{\lambda_3^3}\right),\mathbb{E}\left(U^{(1)}U^{(2)}\right)=\frac{1}{\lambda_1\lambda_2}+\frac{1}{\lambda_1\lambda_3}+\frac{1}{\lambda_2\lambda_3}+\frac{2}{\lambda_3^2}.$$

If we consider $\theta_i=\frac{1}{\lambda_i}$ then for $i\in\{1,2,3\}$, we have:

$$m^{(i)}=a_i+\lambda\ (\theta_i+\theta_3),x^{(i)}:=\frac{1}{2}\left(\mu_2^{(i)}-m^{(i)}\right)=\lambda\ \left(\theta_i^2+\theta_i\theta_3+\theta_3^2\right),$$
$$y^{(i)}:=\frac{1}{6}\left(\mu_3^{(i)}-3\mu_2^{(i)}+m^{(i)}\right)=\lambda\ \left(\theta_i^3+\theta_i^2\theta_3+\theta_i\theta_3^2+\theta_3^3\right),c_{1,2}=\lambda\left(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3+2\theta_3^2\right).$$

Let's note $y^{(i)}-\theta_i x^{(i)}-\lambda\theta_3^3=0$ then the following function is minimized:

$$\sum_{i=1}^{2}\left(x^{(i)}-\lambda\ \left(\theta_i^2+\theta_i\theta_3+\theta_3^2\right)\right)^2+\sum_{i=1}^{2}\left(y^{(i)}-\theta_i x^{(i)}-\lambda\theta_3^3\right)^2+\left(c_{1,2}-\lambda\left(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3+2\theta_3^2\right)\right)^2.$$

This give an estimation $\left(\hat{\lambda},\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3\right)$ of $(\lambda,\theta_1,\theta_2,\theta_3)$, then for $i=1,2$ et $j=1,2,3$:

$$\hat{a}_i=m^{(i)}-\hat{\lambda}\ \left(\hat{\theta}_i+\hat{\theta}_3\right),\hat{\lambda}_j=\frac{1}{\hat{\theta}_j}$$

In Table bellow, a numerical example is given for 1000 paths at times 1, 2, 3, ..., 100, and the confidence intervals are calculated with 500 simulations.

parameter	a_1	a_2	λ	λ_1	λ_2	λ_3
true value	1	2	5	2	1	3
mean	0.97749	1.9687	5.0558	2.0102	1.0057	3.0189
standard déviation	0.12285	0.16467	0.27253	0.058268	0.028479	0.1352
CI 95%	[0.96672,0.98826]	[1.9543,1.9831]	[5.0319,5.0796]	[2.0051,2.0153]	[1.0032,1.0082]	[3.007,3.0307]

4 Reliability Assessment

4.1 General expression in case of any coherent system structure

We consider a system with a coherent structure Φ and n components whose degradation is modeled by $(X_t)_{t\geq 0}$. For each component i, the failure threshold is noted L_i. $\bar{\mathcal{D}}$ is the failure area, i.e.

$$\bar{\mathcal{D}}=\left\{x\in\mathbb{R}_+^n:\Phi(x)=1\right\}.$$

Since Φ is increasing, if $x\leq y$ and $\Phi(x)=1$, then $\Phi(y)=1$. Hence if $x\leq y$ and $x\in\bar{\mathcal{D}}$, then $y\in\bar{\mathcal{D}}$: the failure area $\bar{\mathcal{D}}$ is an upper set.

Proposition 4.1 *The reliability of the system $R(t)$ is equal to:* $R(t)=\mathbb{E}\left[\int_{\bar{\mathcal{D}}^C}\left(\prod_{i=1}^n f^{(\Gamma)}_{a_i t+Y_t^{(i)}}(z_i)\right)dz\right]$ *where $f_\theta^{(\Gamma)}$ is the Gamma probability density fonction with shape θ and scale 1.*

Proof:
$R(t)=\mathbb{P}\left(X_t\notin\bar{\mathcal{D}}\right)=\mathbb{E}\left[\mathbb{E}\left(\mathbf{1}_{Z_{at+Y_t}\notin\bar{\mathcal{D}}}|Y_t\right)\right]=\mathbb{E}\left[G\left(Y_t\right)\right]$

with:

$$G(x) = \mathbb{E}\left(\mathbf{1}_{Z_{at+x} \notin \bar{\mathcal{D}}}\right) = \mathbb{P}\left(Z_{at+x} \notin \bar{\mathcal{D}}\right) = \int_{\bar{\mathcal{D}}^C} \left(\prod_{i=1}^{n} f^{(\Gamma)}_{a_i t + x_i}(z_i)\right) dz$$

□

Apart from particular cases, the law of Y_t is not tractable analytically. The goal in the following is to show some general properties of $R(t)$ based on the above expression.

4.2 Impact of the shocks frequency

The main goal here is to study the impact of the shocks frequency on the system lifetime.

Proposition 4.2 *Consider two systems S and $\tilde{S}$ with the same structure, and the same parameters except the ones of the Poisson processes that triggers the shock arrivals. We note them respectively λ and $\tilde{\lambda}$ for S and $\tilde{S}$. We have for example: $\lambda \leq \tilde{\lambda}$. If we note $R(t)$ and $\tilde{R}(t)$ the associated reliabilities, then $R(t) \geq \tilde{R}(t)$ for any $t \geq 0$ and the lifetime of $\tilde{S}$ is stochastically lower than the one of S .*

Proof:

Since $\lambda \leq \tilde{\lambda}$, $N_t \prec_{sto} \tilde{N}_t$, where $\prec_{sto}$ states the usual stochastic order [2]. We can deduce from [4] that:

$$Y_t = \sum_{i=1}^{N_t} U_i \prec_{sto} \tilde{Y}_t = \sum_{i=1}^{\tilde{N}_t} U_i,$$

Moreover we have: $F_{Z_{at+Y_t}}(z) = \int_{\mathbb{R}_+^n} \prod_{i=1}^n F^{(\Gamma)}_{a_i t + y_i}(z_i) \mathbb{P}_{Y_t}(dy)$ where $F_{Z_{at+Y_t}}$ denote the cumulative density functions of Z_{at+Y_t} and $F^{(\Gamma)}_{a_i t + y_i}$ is the c.d.f. of a Gamma r.v. with shape $a_i t + y_i$ and scale 1.
Let us note R_{y_i} a random variable with a cumulative density function $F^{(\Gamma)}_{a_i t + y_i}$. If $x_i \leq y_i$, then $R_{x_i} \prec_{sto} R_{y_i}$. Since $Y_t \prec_{sto} \tilde{Y}_t$, we can deduce from Thm 6B18 of [4] that $Z_{at+Y_t} \prec_{sto} Z_{at+\tilde{Y}_t}$. At last, $\bar{\mathcal{D}}$ is an upper set, so we know that $\mathbb{P}\left(Z_{at+Y_t} \in \bar{\mathcal{D}}\right) \leq \mathbb{P}\left(Z_{at+\tilde{Y}_t} \in \bar{\mathcal{D}}\right)$ and that $R(t) = \mathbb{P}\left(Z_{at+Y_t} \notin \bar{\mathcal{D}}\right) \geq \tilde{R}(t)$. □

This proposition means that the lifetime of the system stochastically decreases when the shock frequency increases.

4.3 Impact of the dependence between the shocks increments

The main goal here is to study the impact of the dependence between the shocks increments $U^{(j)}$ on the system lifetime. Consider two systems S and $\tilde{S}$ with the same structure, and the same parameters except that the dependence between the increments $\tilde{U}^{(j)}$ is higher than the dependence between the increments $U^{(j)}$. The marginal laws of the increments $\tilde{U}^{(j)}$ and $U^{(j)}$ are the same. This hypothesis can be formalized by the [4], [2] :

- "lower orthant order", where $\tilde{U} \prec_{l.o.} U$ if and only if $F_U \leq F_{\tilde{U}}$,
- "upper orthant order", where $U \prec_{u.o.} \tilde{U}$ if and only if $\bar{F}_U \leq \bar{F}_{\tilde{U}}$,
- "positive quadrant dependence (PQD) order" where $U \prec_{PQD} \tilde{U}$ if and only if ($\tilde{U} \prec_{l.o.} U$ and $U \prec_{u.o.} \tilde{U}$).

Proposition 4.3 *Considering the previous notations, the following results hold:*

1. *If $\tilde{U} \prec_{l.o.} U$ and if the system is a series structure then $R(t) \leq \tilde{R}(t)$ for any $t \geq 0$. Hence, for a series structure, the lifetime is stochastically higher as the dependence between the components increases.*

2. *If $U \prec_{u.o.} \tilde{U}$ and if the system is a parallel structure, then $R(t) \geq \tilde{R}(t)$ for any $t \geq 0$. Hence, for a parallel structure, the lifetime is stochastically higher as the dependence between the components is decreases.*

3. *If $U \prec_{PQD} \tilde{U}$, then $R(t) \leq \tilde{R}(t)$ for a series structure and $R(t) \geq \tilde{R}(t)$ for a parallel one.*

Proof:

1. Considering that $U_i \succ_{l.o.} \tilde{U}_i$ for any $i \geq 1$ and that N_t and $\tilde{U}_i$ are independent, we know according to Thm 6.G.7 in[4] that:
$$Y_t = \sum_{i=1}^{N_t} U_i \succ_{l.o.} \tilde{Y}_t = \sum_{i=1}^{N_t} \tilde{U}_i.$$
Moreover, $F_{Z_{at+Y_t}}(z) = \mathbb{E}\left(\prod_{i=1}^{n} \varphi_i\left(Y_t^{(i)}\right)\right)$ with $\varphi_i(y_i) = F^{(\Gamma)}_{a_i t+y_i}(z_i)$. According to the previous proof, we know that $\varphi_i(y_i)$ is decreasing as a function of y_i (since R_{y_i} with c.d.f. $F^{(\Gamma)}_{a_i t+y_i}$ stochastically increases with y_i). According to Thm 6.G.1(b) in [4], we can deduce that:
$$F_{Z_{at+Y_t}}(z) = \mathbb{E}\left(\prod_{i=1}^{n} \varphi_i\left(Y_t^{(i)}\right)\right) \leq F_{Z_{at+\tilde{Y}_t}}(z) = \mathbb{E}\left(\prod_{i=1}^{n} \varphi_i\left(\tilde{Y}_t^{(i)}\right)\right)$$
and that:
$$R(t) = \mathbb{P}\left(Z_{at+Y_t} \in \bar{\mathcal{D}}^c\right) = F_{Z_{at+Y_t}}(L) \leq \tilde{R}(t)$$
since for a series system, we have $\bar{\mathcal{D}}^c = [0, L_1[\times \cdots \times [0, L_n[$.

2. Considering that $U_i \prec_{u.o.} \tilde{U}_i$ for any $i \geq 1$, we show in the same way: $Y_t \prec_{u.o.} \tilde{Y}_t$. Moreover $\bar{F}_{Z_{at+Y_t}}(z) = \mathbb{E}\left(\prod_{i=1}^{n} \psi_i\left(Y_t^{(i)}\right)\right)$ with $\psi_i(y_i) = \bar{F}^{(\Gamma)}_{a_i t+y_i}(z_i)$, where ψ_i increases as a function of y_i. According to Thm 6.G.1(a) in [4], we deduce that:
$$\bar{F}_{Z_{at+Y_t}}(z) = \mathbb{E}\left(\prod_{i=1}^{n} \psi_i\left(Y_t^{(i)}\right)\right) \leq \bar{F}_{Z_{at+\tilde{Y}_t}}(z)$$
and that
$$1 - R(t) = \mathbb{P}\left(Z_{at+Y_t} \in \bar{\mathcal{D}}\right) = \bar{F}_{Z_{at+Y_t}}(L) \leq 1 - \tilde{R}(t)$$
since for a parallel structure: $\bar{\mathcal{D}} = [L_1, \infty[\times \cdots \times [L_n, \infty[$.

3. If $U \prec_{PQD} \tilde{U}$, then $\tilde{U} \prec_{l.o.} U$ and $U \prec_{u.o.} \tilde{U}$.

□

4.4 Numerical illustration

4.4.1 Model of dependence for a three units system

Let us consider a system of 3 units, with one in series (C_1), and two in parallel (C_2 and C_3), see Figure 1.

[Fig. 1 about here.]

Then:
$$\bar{\mathcal{D}} = \left([L_1, \infty[\times \mathbb{R}_+^2\right) \cup \{\mathbb{R}_+ \times [L_2, \infty[\times [L_3, \infty[\}$$
and
$$R(t) = \mathbb{E}\left[F^{(\Gamma)}_{a_1 t+Y_t^{(1)}}(L_1)\left(1 - \bar{F}^{(\Gamma)}_{a_2 t+Y_t^{(2)}}(L_2)\,\bar{F}^{(\Gamma)}_{a_3 t+Y_t^{(3)}}(L_3)\right)\right]$$
To focus on the dependence, lets take $a_i = 0$. Concerning the dependence, we choose the following model: for $i \in \{1, 2, 3\}$ and $U_i = V_i + V_4$. The laws of the variables V_i is $\Gamma(\alpha_i, 1)$ for $i \in \{1, \cdots 4\}$ and are independent. So for $i = 1 : 3$, the law of V_i is $\Gamma(A_i, 1)$, with $A_i = \alpha_i + \alpha_4$ and the dependence is directly controlled by the parameter α_4.

4.4.2 Impact of the dependancies

The aim is to change the dependence but keeping the same marginal laws (A_i is fixed for $i = 1:3$). As the dependence is controlled by the parameter α_4, let it vary between 0 and $\min(A_i, i = 1:3)$, and consider $\alpha_i = A_i - \alpha_4$.

Figures 2 and 3 show the evolution of $F_U(u)$ and $\bar{F}_U(u)$ as a function of α_4 with $A_i = 1$, for $i = 1:3$ and for $u = (1,1,1)$. When α_4 is increasing, $F_U(u)$ and $\bar{F}_U(u)$ are increasing, which means that the dependence increases according to $\prec_{l.o.}$ and $\prec_{u.o.}$ with α_4, and so increases according to $\prec_{PQD}$.

[Fig. 2 about here.]

Let now take $\lambda = 10$, $\alpha_4 = [0:0.1:1]$ while keeping $A_i = 1$, for $i = 1:3$. The evolution of $R(t_0)$ as a function of α_4 is considered for $t_0 = 10$. Two cases are studied and depicted respectively in Figures 4 and 5: case 1 corresponds to $L = [1000, 100, 100]$ and case 2 to $L = [100, 10, 100]$. In case 1, $R(t_0)$ is decreasing with the dependence, whereas in the second one, it is increasing. A possible interpretation is the following one: in case 1, unit C_1 is much more reliable than the other ones such that the system behavior is closed to a parallel structure, and in case 2, unit 2 is much less reliable than the other ones, such that the system behavior is closed to a series structure. From a more general point of view, it can be noticed that when there is neither pure series nor pure parallel structure, it is impossible to exhibit general rules concerning the impact of the dependence on the system reliability.

Finally the evolution of $R(t_0)$ is given in Figure 6 for λ varying from 5 to 10 and for $A_i = 1$, for $i = 1:3$, $\alpha_4 = 0.5$, $L = [100, 100, 100]$ and $t0 = 10$. It shows the decreasing of the system reliability as the shocks frequency increases as stated in Prop. 4.2.

[Fig. 3 about here.]

5 Conclusion

The proposed model allows to consider stochastic dependencies in a multi-unit context with tractable calculation for the reliability. Further work is necessary to investigate some different topics: the correct use of the additional equation in the estimation step, the non-periodic inspection policy, the calculation of the remaining useful lifetime and the use of the above results in a maintenance context. At last, we are looking for more precise applications with suitable form of dependencies to enhance the interest of the proposed model.

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List of Figures

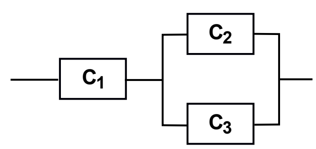

Fig. 1: Structure of the three-unit system.

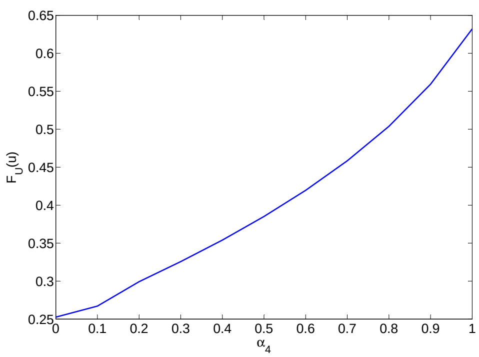

Fig. 2: Evolution of $F_U\left(u\right)$ with respect of α_4.

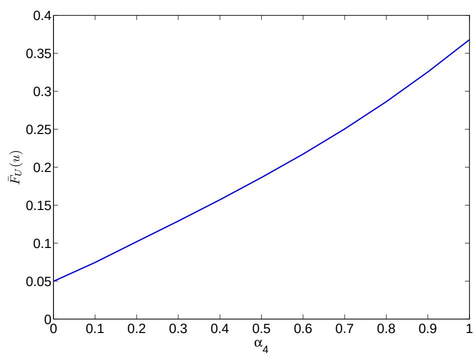

Fig. 3: Evolution of $\bar{F}_U\left(u\right)$ with respect of α_4.

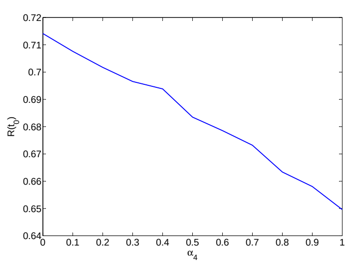

Fig. 4: Evolution of $R\left(t_0\right)$ with respect of α_4, case 1.

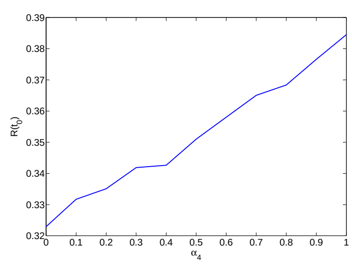

Fig. 5: Evolution of $R\left(t_0\right)$ with respect α_4, case 2.

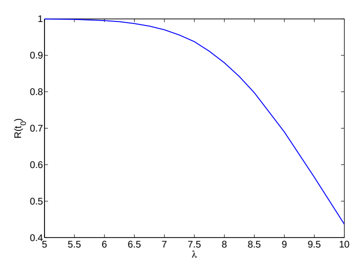

Fig. 6: Evolution of $R\left(t_0\right)$ with respect of λ.